Continuum Models and Discrete Systems (CMDS9)
Proceedings of the 9th International Symposium
June 29–July 3, 1998, Istanbul, Turkey. Editors E. Inan & K. Z. Markov

WAVES IN PIEZOELECTRIC RANDOM MEDIA

F. J. SABINA

Department of Applied Mathematics and Theoretical Physics,
Cambridge University, Silver Street, Cambridge CB3 9EW, Great Britain
On leave from:
Instituto de Investigaciones en Matemáticas Aplicadas y en Sistemas,
Universidad Nacional Autónoma de México,
Apartado Postal 20–726, 01000 México, D. F., México

A new theory has been developed for the analysis of the overall properties and the propagation of waves though inhomogeneous piezoelectric materials with randomly distributed microstructure. The self-consistent method has been used by means of a very simple instrumentation, which allows the prediction of resonance phenomena for wavelengths comparable to the diameter of the inclusions. Here the problem of predicting the overall properties for 0–3 connectivity piezocomposites and wave propagation through a piezoelectric matrix containing piezoelectric ellipsoidal inclusions, randomly positioned, but with a fixed orientation are studied. From the overall elastic, piezoelectric and dielectric properties of the piezocomposite, an eigenvalue problem yields the phase speed and the attenuation of the waves as a function of the frequency of the waves in a equivalent homogeneous, but attenuative, piezoelectric composite. The results show a characteristic resonance phenomena which depends on the azimuthal angle and type of wave.

Keywords. Micromechanics, Random media, Composites, Overall properties, Wave attenuation, Wave dispersion, Piezoelectricity, Electromechanical

1991/95 Math. Subject Class. 73B27, 73B35, 73B40, 73D25, 73R05, 73S10

1 Introduction

Composite piezoelectric materials, characterized by coupled effects between strain and electric field, have recently been found useful in the design of new transducers, which were tailored to meet certain criteria in applications like ultrasound medical imaging, nondestructive testing and underwater acoustics [1]. The study of waves in composites containing embedded inclusions in a matrix for uncoupled elasticity of [2], referred to as I, is further extended to include coupled effects like piezoelectricity. This composite has been studied in the static regime [3–4].

By means of the present approach, useful results are obtained up to frequencies which corresponds to half-wavelength waves and diameter inclusions of comparable size. Namely, certain "resonance" effects which correspond to relatively large variations in phase speeds, and large atenuation of the mean wave, through the transfer of energy into coherent motion of single and multiple scattering.

The general formulae that appear in I is applicable to the present study by a simple, but effective identification of the relevant quantities that appear in the uncoupled elastic problem and its piezoelectric counterpart here. The prefix I before an equation number refers to that of I.

2 Self-consistent equations

In this paper the study of waves in piezocomposites starts with equations (I.22) and (I.23). As a prelude towards this aim, the piezoelectric composite that is considered here consists of a matrix with material properties given by the fourth, third and second-order tensors of elastic moduli $c \equiv c^E_{n+1}$ at constant electric field E, piezoelectric constants e_{n+1} and dielectric permittivity $\epsilon \equiv \epsilon^S_{n+1}$ at constant strain S, respectively, and the mass density ρ_{n+1}, in which are embedded n different types of inclusions, positioned randomly without overlapping. An inclusion of type r has corresponding properties denoted by c_r, e_r, ϵ_r and ρ_r. It occupies a domain $\boldsymbol{x}' + \Omega_r$, where Ω_r contains the origin. Each inclusion of type r thus has the same shape, size and orientation. The composite will be taken to occupy a domain Ω, large enough to contain many inclusions.

In the absence of body forces and charge densities, the dynamic response of the composite under isothermal conditions is governed by the equations of motion [5]

$$T_{ij,j} = \dot{p}_i, \tag{2.1}$$
$$D_{i,i} = 0, \tag{2.2}$$

where the second-order stress tensor T, electric displacement vector D and momentum density vector p are related to the second-order strain tensor S, electric field potential ϕ and velocity vector v through the constitutive relations

$$T_{ij} = c_{ijkl}S_{kl} + e_{kij}\phi_{,k}, \tag{2.3}$$

$$D_i = e_{ikl} S_{kl} - \epsilon_{ik} \phi_{,k}, \tag{2.4}$$
$$p_i = \rho v_i, \tag{2.5}$$

where the summation convention of repeated indices is understood and the indices i, j, k and l take the values 1, 2 and 3; it is noted that the material properties appear explicitly as the coefficients of S, ϕ and v, with strain S and velocity v related to displacement vector u through

$$S_{ij} = (u_{i,j} + u_{j,i})/2, \tag{2.6}$$
$$v_i = \dot{u}_i, \tag{2.7}$$

to this point the comma and dot notations are interpreted as $u_{i,j} \equiv \partial u_i / \partial x_j$ and $\dot{u}_i \equiv \partial u_i / \partial t$, respectively.

The relations (2.3), (2.4) and (2.5) are written in another symbolic form as follows

$$\mathcal{T} = \mathcal{C}\mathcal{S}, \tag{2.8}$$

where

$$\mathcal{T} = \begin{pmatrix} T & D & p \end{pmatrix}, \tag{2.9}$$

$$\mathcal{C} = \begin{pmatrix} c & e^t & 0 \\ e & -\epsilon & 0 \\ 0 & 0 & \rho \end{pmatrix}, \tag{2.10}$$

$$\mathcal{S} = \begin{pmatrix} S \\ \mathrm{grad}\,\phi \\ v \end{pmatrix}; \tag{2.11}$$

where e^t denotes the transpose of e. In this notation the "matrix" $\mathcal{C}$ has to be considered as a linear operator which transforms the tensor triad (T, D, p) into another tensor triad $(S, \mathrm{grad}\,\phi, v)$. Then, (2.8) can be viewed as a generalized Hooke's law and $\mathcal{T}$, $\mathcal{C}$ and $\mathcal{S}$ as generalized stress, elastic moduli and strain, respectively. With this in mind, the self-consistent requirements (I.22) translated into the notation for the piezoelectric composite become

$$\langle \mathcal{T} \rangle = \mathcal{C}_0 \langle \mathcal{S} \rangle, \quad \text{with} \quad \langle \mathcal{S} \rangle = \mathcal{S}_0, \tag{2.12}$$

where $\mathcal{S}_0$ is obtained from the incident wave

$$u_0(x,t) = m \exp \mathrm{i}(k_0 n.x - \omega t), \tag{2.13}$$
$$\phi_0(x,t) = \varphi \exp \mathrm{i}(k_0 n.x), \tag{2.14}$$

which propagates in a reference material of properties $\mathcal{C}_0$; here, m is the displacement polarization vector, φ is the electric potential amplitude, n is the unit vector defining the direction of the wave and k_0 is the wavenumber. Note that the electric potential ϕ_0 does not depend on time, but has a spatial harmonic variation.

The self-consistent conditions follow analogously from (I.23) to become

$$\mathcal{C}_0\mathcal{S}_0 = \left\{ \mathcal{C}_{n+1} + \sum_{r=1}^{n} f_r h_r(k_0 n) h_r(-k_0 n)(\mathcal{C}_r - \mathcal{C}_{n+1})[I + \bar{\mathcal{G}}_{\mathcal{X}}^{(r)}(\mathcal{C}_r - \mathcal{C}_0)]^{-1} \right\} \mathcal{S}_0, \tag{2.15}$$

where f_r is the volume fraction occupied by material r. The appropiate piezoelectric eigenvalue problem is

$$\left\{ k_0^2 \left[(c_0)_{ijkl} n_j n_l + [(e_0)_{lij}(e_0)_{mkn} n_m n_l n_j n_n / (\epsilon_0)_{rs} n_r n_s] \right] - \omega^2 (\rho_0)_{ik} \right\} m_k = 0, \tag{2.16}$$

where ω is the circular frequency. Note that, as in the uncoupled problem [2], the effective density is a second-order tensor. It follows immediately from the resulting Furthermore,

$$\begin{aligned} h_r(k_0 n) &= \frac{1}{|\Omega_r|} \int_{\Omega_r} \exp(\mathrm{i} k_0 n.x) \mathrm{d}x, && (2.17) \\ \bar{\mathcal{G}}_{\mathcal{X}}^{(r)} &= \frac{1}{|\Omega_r|} \int_{\Omega_r} \mathcal{G}_{\mathcal{X}} \mathrm{d}x, && (2.18) \end{aligned}$$

where the bar represents mean value over the inclusion, the index r refers to the inclusion type, and

$$\mathcal{G}_{\mathcal{X}} = \begin{pmatrix} P_x & N_x & M_x \\ P_{,x} & N_{,x} & M_{,x} \\ P_{,t} & N_{,t} & M_{,t} \end{pmatrix}; \tag{2.19}$$

the entries in (2.19) are (generalized) functions which are obtained from differentiating the free space Green's function G corresponding to piezoelectricity [6]; in component form, they are given as

$$\begin{aligned} (P_x)_{ijkl} &= -G_{ik,lj}|_{(ij)(kl)}, && (2.20) \\ (P_{,x})_{4lij} &= -G_{4i,jl}|_{(ij)}, && (2.21) \\ (P_{,t})_{kij} &= \mathrm{i}\omega G_{ki,j}|_{(ij)}, && (2.22) \\ (N_x)_{kl4j} &= G_{k4,lj}|_{(kl)}, && (2.23) \\ (N_{,x})_{4l4j} &= G_{44,jl}, && (2.24) \end{aligned}$$

$$(N_{,t})_{k4j} = -\mathrm{i}\omega G_{k4,j}, \tag{2.25}$$
$$(M_{x})_{kli} = -\mathrm{i}\omega G_{ki,l}|_{(kl)}, \tag{2.26}$$
$$(M_{,x})_{4li} = -\mathrm{i}\omega G_{4i,l}, \tag{2.27}$$
$$(M_{,t})_{ki} = -\omega^2 G_{ki}, \tag{2.28}$$

Many choices of $\mathcal{C}_0$ are possible as a solution of (2.15), with (2.16). Strictly, (2.15) have only to be satisfied for the particular $\mathcal{S}_0$, of present concern. In [7] it was proposed, however, that (2.15) should be relaxed by requiring the operators on either side to agree, cancelling, in effect, the fields $\mathcal{S}_0$ upon which they operate. The fields do still appear implicitly, through the choice of wavenumber k_0 that has to be compatible with (2.16).

3 Aligned spheroids

A two phase material is now considered with a single population of spheroidal inclusions whose domain Ω_1 is the set

$$\Omega_1 = \{x : x_1^2 + x_2^2 + x_3^2/\delta^2 < a^2\}, \tag{3.1}$$

where δ is the aspect ratio and a is the semi-major axis. Thus $n = 1$, and the self-consistent equations (2.15), omitting the factor $\mathcal{S}_0$, become

$$\mathcal{C}_0 = \mathcal{C}_2 + f_1 h_1(k_0 n) h_1(-k_0 n)(\mathcal{C}_1 - \mathcal{C}_2)[I + \bar{\mathcal{G}}^{(1)}_{\mathcal{X}}(\mathcal{C}_1 - \mathcal{C}_0)]^{-1}. \tag{3.2}$$

The only distinguished direction in equations (3.2) is the axis of symmetry, Ox_3, of the spheroids. It is consistent, therefore, to take the entries of $\mathcal{C}_0$ to be tensors of the relevant order with transversely isotropic symmetry. A generalization of Hill's notation [8] for the elastic moduli tensor c allows the constitutive relations (2.8), and thus for $\mathcal{C}$, to be written, in an expanded form, for the piezoelectric material as follows [9]:

$$(T_{11} + T_{22})/2 = k(S_{11} + S_{22}) + l' S_{33} + b'\phi_{,3}, \tag{3.3}$$
$$(T_{11} - T_{22}) = 2m(S_{11} - S_{22}), \tag{3.4}$$
$$T_{33} = l(S_{11} + S_{22}) + nS_{33} + c'\phi_{,3}, \tag{3.5}$$
$$T_{23} = 2pS_{23} + a'\phi_{,2}, \tag{3.6}$$
$$T_{31} = 2pS_{31} + a'\phi_{,1}, \tag{3.7}$$
$$T_{12} = 2mS_{12}, \tag{3.8}$$
$$D_1 = 2aS_{13} + v\phi_{,1}, \tag{3.9}$$
$$D_2 = 2aS_{23} + v\phi_{,2}, \tag{3.10}$$

$$D_3 = b(S_{11} + S_{22}) + cS_{33} + w\phi_{,3}, \tag{3.11}$$
$$p_1 = -\mathrm{i}\omega\rho_I u_1, \tag{3.12}$$
$$p_3 = -\mathrm{i}\omega\rho_{III} u_3. \tag{3.13}$$

Note that the off-diagonal terms are distinguished by either a prime or its absence. This allowance has been made due to the fact that the diagonal symmetry is lost when two transversely isotropic tensors are multiplied such as in (3.2). Furthermore, Hill's symbolic notation is now generalized for each of the entries of $\mathcal{C}_0$ so that

$$c_0 = (2k_0, l_0, l'_0, n_0, 2m_0, 2p_0), \tag{3.14}$$
$$\rho_0 = (\rho_I, \rho_{III}), \tag{3.15}$$
$$e_0 = (2a_0, b_0, c_0, a'_0, b'_0, c'_0), \tag{3.16}$$
$$-\epsilon_0 = (v_0, w_0). \tag{3.17}$$

The tensor equation (3.2) is then resolved in a much reduced number of equations, namely, the sixteen component equation

$$Z = f(Z, \omega), \tag{3.18}$$

where

$$Z = (2k_0, l_0, l'_0, n_0, 2m_0, 2p_0, 2a_0, b_0, c_0, a'_0, b'_0, c'_0, v_0, w_0, \rho_I, \rho_{III}). \tag{3.19}$$

Although the possible solutions of (3.18) now lie in a higher dimension space, the solution in the end must satisfy the symmetry requirements of the problem. Equations (3.18) were obtained by means of a simple algebraic structure that is easily obtained in the same manner as Hill derived the one for c. These results were exploited in [2]. Thus, if Z_1 and Z_2 are like (3.19), then $Z_1 + Z_2$, $Z_1 Z_2$ and Z_1^{-1} are also like (3.19). Also, the functions $\mathcal{G}_{\mathcal{X}}$ can be constructed as in [2] and the outcome $\bar{\mathcal{G}}_{\mathcal{X}}^{(1)}$ are expressible in the form (3.19). Each one of them requires a one dimensional integration which must be performed numerically (Cf. [2]) The above expressions will appear in a forthcoming paper.

Before proceeding to numerical results, it is noted first that, at low frequency ($\omega \to 0$), the "third" equation of (3.2) then implies that $\rho_0 = f_1\rho_1 + f_2\rho_2$; that is, the effective density is the mean density, as expected. Also, in this limit the equations (3.2) reduce to those obtained directly in the static case by [3],

4 Example

These equations were solved by iteration, for a range of circular frequencies $\omega = \omega_i$, $i = 0, 1, 2, \cdots$, with $\omega_0 = 0$ and $\{\omega_i\}$ an increasing sequence. A porous PZT–7A ceramic is now considered with spherical pores of radius a. The material properties are taken from [3]. The dispersion of a quasi P-wave with azimuthal angle at $\pi/4$ is shown in Fig. 1. Dimensionless wave speed v/v_2 is plotted against normalized wavenumber k_2a, where v is the wave speed in the composite and v_2 is the P-wave speed of the matrix and $k_2$is its wavenumber. Two plots are shown, one for an unpoled ceramic (dashes) and the other for a poled one (continuous). Note that the piezoelectricy has the effect of increasing the stiffness of the ceramic. Thus the waves travels faster in the poled PZT–7A for very long wavelengths up the perimeter of a void's great circle.

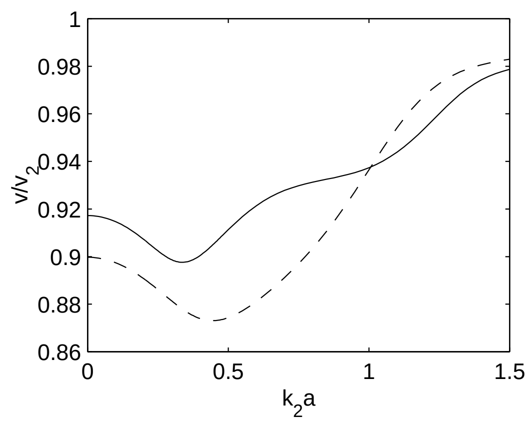

Fig. 1: Dispersion of a quasi-P wave in a 10% porous PZT–7A ceramic. (a) Dimensionless speed v/v_2 versus normalized wavenumber k_2a. Poled ceramic (continuous line), unpoled ceramic (dash line).

Acknowledgements. FJS would like to thank the John Simon Guggenheim Memorial Foundation for the award of one of its Fellowships and support from CoNaCyT, project number 3229P-E9607 is gratefully acknowledged. I am also indebted and thankful for the facilities provided by the Department of Applied Mathematics and Theoretical Physics, Cambridge University during my sabbatical leave. Finally, I thank Reinaldo Rodríguez for his encouragement in this and interest in other piezoelectric problems.

References

1. W. A. Smith, In *Proc. 1990 IEEE 7th Int. Symp. Ferroelec.*, IEEE, New York, 1991, p. 145.

2. F. J. Sabina, V. P. Smyshlyaev and J. R. Willis, *J. Mech. Phys. Solids* **41** (1993) 1573.
3. M. L. Dunn and M. Taya, *Int. J. Solids Struct.* **30** (1993) 161.
4. V. M. Levin, In *Continuum Models and Discrete Systems*, ed. K. Z. Markov, World Scientific Publishing Co., 1996, p. 225.
5. T. Ikeda, *Fundamentals of Piezoelectricity*, Oxford University Press, Oxford, 1990.
6. A. N. Norris, *Proc. R. Soc. Lond. A* **447** (1994) 175.
7. F. J. Sabina and J. R. Willis, *Wave Motion* **10** (1988) 127.
8. R. Hill, *J. Mech. Phys. Solids* **13** (1965) 189.
9. K. Schulgasser, *J. Mech. Phys. Solids* **40** (1992) .