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Symmetry-Breaking Convective Dynamos in Spherical Shells

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Summary. The convective dynamo is the generation of a magnetic field by the convective motion of an electrically conducting fluid. We assume a spherical domain and spherically invariant basic equations and boundary conditions. The initial state of rest is then spherically symmetric. A first instability leads to purely convective flows, the pattern of which is selected according to the known classification of $O(3)$-symmetry-breaking bifurcation theory. A second instability can then lead to the dynamo effect. Computing this instability is now a purely numerical problem, because the convective flow is known only by its numerical approximation. However, since the convective flow can still possess a nontrivial symmetry group G_0, this is again a symmetry-breaking bifurcation problem. After having determined numerically the critical linear magnetic modes, we determine the action of G_0 in the space of these critical modes. Applying methods of equivariant bifurcation theory, we can classify the pattern selection rules in the dynamo bifurcation. We consider various aspect ratios of the spherical fluid domain, corresponding to different convective patterns, and we are able to describe the symmetry and generic properties of the bifurcated magnetic fields.

1. Introduction

A convective dynamo is a mechanism by which a magnetic field is generated and self-sustained by the motion of an electrically conducting fluid in a certain domain. This motion can itself be produced by various forces like, for example, thermal convection. In that case, the model equations are the Navier-Stokes equations coupled with the heat equation and the Maxwell equations. The current produced by the fluid motion tends to generate a magnetic field; however, this effect is counterbalanced by ohmic dissipation. On the other hand, the magnetic field itself acts on the motion through the Lorentz force.

This instability leads therefore to a bifurcation problem for a magnetohydrodynamic system. It is a known fact that, if the fluid flow is not too simple (in a certain sense which we need not make precise here), an instability can indeed occur that allows for growing magnetic disturbances (see, e.g., Proctor and Gilbert [22] for details and bibliography). It should also be noted that this problem has interested specialists for several decades because it is based on the study of the mechanisms by which planetary magnetic fields can exist over long periods of times (in the geological scale) and have such complex behavior as, for example, the Earth's poles' reversals (on time scales of about 10^5 years). In this context, it is natural to consider a self-gravitating fluid domain bounded by concentric spheres. It is well-known that the model equations are then *spherically invariant*. However, it is also well-known that inertial forces due to the rotation of the Earth around its axis have a strong effect on the convective motion and hence on the dynamo. A reasonable model for the Earth's dynamo should therefore take such effects into account. This has been done in recent papers on the geodynamo (e.g., Zhang and Busse [29], Sarson and Gubbins [24], Whicht and Busse [27]).

In the present paper, we are interested in the dynamo generated by pure convection, i.e., when the domain is fixed. This would clearly be a strong limitation if our aim were primarily to study a model for the geodynamo. Our main objective in this work is to study *pattern selection properties* of the convective dynamo in the case when the high degree of symmetry of the basic system (full orthogonal group $O(3)$) induces a high degeneracy in the bifurcation problem. An analysis of the symmetry properties of the dynamo problem has been done recently by Gubbins and Zhang [13]. Primarily interested by the geodynamo, these authors have considered the case of a strongly rotating domain, which admits a much smaller symmetry group than the nonrotating one (this group is isomorphic to $SO(2) \times Z_2$ instead of $O(3)$ in the case without rotation). In this case, the pattern selection rules and the dynamical behavior are rather straightforward and do not require the machinery of equivariant bifurcation methods (see [11] for an introductory exposition of this theory). We show in this paper that a suitable application of general theorems about bifurcation in the presence of symmetry allows us to deduce relevant and interesting properties of the dynamo bifurcation in a system with initial spherical symmetry, at relatively low cost (that is, after a linear stability analysis of the pure convective flow). The universal properties of pattern selection rule and dynamical behavior of bifurcation problems with $O(3)$ symmetry have strong consequences for the dynamo bifurcation. As an example, let us comment on the following result: In a system with continuous symmetry, primary branches of equilibria (that is, the convective flows bifurcating from a trivial state which, in our case, is the pure conduction state) are not isolated, but rather are a continuum of equilibria forced by the symmetry (what is called a "group orbit" of equilibria). It is known that a secondary steady-state bifurcation arising from this branch can in fact lead to *time-dependent* solutions, which assume the form of *rotating waves*. For this, a generic condition needs be satisfied (see the introductory part of Section 3). It turns out that this condition is not fulfilled in the cases we have studied. The obstruction to time dependence comes from an *additional symmetry* in the model, namely the symmetry $\mathbf{b} \mapsto -\mathbf{b}$ ($\mathbf{b}$ is the magnetic field). Therefore, the dynamo produces "genuine" steady-states, but this is not an obvious fact.

Two basic approaches have been used to study the dynamo instability problem. In the first approach, one starts from a known fluid flow and studies (numerically) its linear stability under magnetic disturbances. This is known as the *kinematic dynamo* problem. There are two limitations to this approach: First, it does not fully solve the bifurcation problem since it only deals with linear stability. Second, the convective flows that are input in the induction equation are usually rather simple models of flows, but are not solutions of the actual hydrodynamic equations (see, however, Gubbins [12] and Dudley and James [7]). The second approach is a direct simulation of the full M.H.D. system. This work was undertaken about 12 years ago and has recently led to impressive results (see, e.g., Glatzmaier [10]). However, running such huge programs on big computers is time consuming, and the results can hardly provide a clear understanding of the mechanism of *pattern selection* that lead to the observed magnetic fields. This approach is more appropriate when attempting to simulate, in as realistic a situation as possible, the large-scale behavior of the Earth's magnetic field over geological time.

Our method in this paper is based on the kinematic dynamo approach. To be more specific: (i) we input flows that are the leading part of the primary branches of pure convective solution to the classical Rayleigh-Bénard convection equations in a spherical shell. This is justified, as we shall see in Section 2, if the magnetic Prandtl number (ratio of the electrical conductivity to the magnetic diffusivity) is small enough. The pattern selection rules and stability for these flows have been studied in detail, notably by Busse [1], Busse and Riahi [2], and Chossat and Giraud [5]. (ii) After the critical parameter values and the critical modes have been determined by the linear stability analysis of the induction equation, an examination of the action of the symmetry group of the pure convective flow on the critical eigenvectors allows us to determine the *symmetry-breaking type* of this bifurcation. We can then apply equivariant bifurcation theory to list the bifurcating solutions, with their symmetries and possible stability property. This part of the analysis is genuinely nonlinear, although we do not explicitly compute the bifurcation equations. It tells us what kind of solutions do typically bifurcate, and what their main properties are.

The paper is organized as follows: In Section 2, we present the mathematical formulation of the problem and a review of basic facts about bifurcation with spherical symmetry. In Section 3, we explain how the dynamo problem can be considered as a secondary bifurcation and we discuss the strategy for determining the symmetry-breaking type of the bifurcation. Then we present the results of our analysis for various kinds of velocity fields. A short description of the numerical method is also included and its efficiency is analyzed.

We have considered the following cases (η is the aspect ratio of the spherical shell):

1. $\eta = 0.1$, in which case the basic flow is axisymmetric with leading spherical harmonics of degree $\ell = 1$.
2. $\eta = 0.3$. In this case, the basic flow can be either axisymmetric or nonaxisymmetric, with leading spherical harmonics of degree $\ell = 2$.
3. $\eta = 0.4$. In this case, the basic axisymmetric flow is always unstable, but there are two other kinds of flows: one with a hexagonal-like symmetry, the other with a tetrahedral symmetry, and each of these can be stable. The corresponding spherical harmonics in this case have degree $\ell = 3$.

The paper ends with a discussion of the results and some concluding remarks in Section 4.

2. Mathematical Formulation of the Problem

2.1. The M.H.D. Equations

Our setting for the model of thermal convection follows [5]. We consider the fluid shell bounded by two spheres, with outer radius r_o and inner radius r_i. The fluid being assumed homogeneous, the gravity field has the form $g^*(r)\mathbf{r}$, with

$$g^*(r) = g_1 + g_0/r^3, \qquad g_0,\ g_1 \geq 0.$$

We also assume a homogeneous distribution of heat sources in the domain and at the inner boundary, so that the temperature gradient in the absence of convection has the form $h^*(r)\mathbf{r}$, with

$$h^*(r) = h_1 + h_0/r^3, \qquad h_0,\ h_1 \geq 0.$$

We assume that inertial forces can be neglected; that is, we consider the domain fixed. In other words, we are looking for dynamos *driven by thermal convection only*.

We introduce nondimensional variables by using r_o, r_o^2/ν, $\epsilon d^2 \nu\kappa^{-1}$, and $\nu(\mu_o \varrho)^{1/2}/r_o$ as units of length, time, temperature, and magnetic field, respectively, where ν, ϵ, κ, μ, and ϱ are the kinematic viscosity, thermal expansivity, thermal diffusivity, magnetic permeability, and density of the fluid. Then the following nondimensional parameters are coming into the equations: the Rayleigh number $Ra = \epsilon\ g(r_o)\ h(r_o)\ r_o^6/\nu\kappa$, the Prandtl number $P_r = \nu/\kappa$, and $\beta = (\mu\sigma\nu)^{-1}$ (inverse of the magnetic Prandtl number, σ being the electrical conductivity). All terms that can be written as gradients have been incorporated in the gradient of pressure $\nabla\Pi$. The domain D is now bounded by concentric spheres of radii $\eta = r_i/r_o$ and 1. We write $\mathbf{u}$ for the velocity field, Θ for the perturbation of the temperature field at pure conduction, and $\mathbf{b}$ for the magnetic field. The equations for $(\mathbf{u}, \Theta, \mathbf{b})$ are

$$\frac{\partial \mathbf{u}}{\partial t} = \triangle\mathbf{u} - \nabla\Pi + \sqrt{Ra}\ \Theta g^*(\mathbf{r}) - \mathbf{u}.\nabla\mathbf{u} - \mathbf{b} \times \nabla \times \mathbf{b}, \tag{1}$$

$$\nabla.\mathbf{u} = 0, \tag{2}$$

$$\frac{\partial \Theta}{\partial t} = P_r^{-1}(\triangle\Theta + \sqrt{Ra}\ h^*(\mathbf{r})\mathbf{u}.\mathbf{r}) - \mathbf{u}.\nabla\Theta, \tag{3}$$

$$\frac{\partial \mathbf{b}}{\partial t} = \beta\triangle\mathbf{b} + \nabla \times (\mathbf{u} \times \mathbf{b}), \tag{4}$$

$$\nabla.\mathbf{b} = 0. \tag{5}$$

The usual Boussinesq approximation has been assumed in (1) to (3), which means that all material properties are regarded as constant with respect to temperature, except the density which is taken into account only in the buoyancy term in (1).

We made the following assumptions: (i) vanishing velocity field and constant temperature at the boundary, i.e.,

$$u_r = u_\theta = u_\varphi = \Theta = 0, \qquad \text{at } r = \eta \text{ and } r = 1; \tag{6}$$

(ii) perfect conductor outside the domain, which means, with the rigid boundary conditions for the velocity field above,

$$b_r = (\nabla \times\ b)_\theta = (\nabla \times\ b)_\varphi = 0 \qquad \text{at } r = \eta \text{ and } r = 1. \tag{7}$$

Note that if these equations were intended to be a model for the geodynamo, then not only should rotation be introduced, but also more realistic boundary conditions should be considered, reflecting the fact that the mantle is a poor conductor, that the core-mantle and fluid core–inner core interfaces are surfaces of transition from fluid to solid, and that the exterior field decays to zero in a vacuum. Our boundary conditions were chosen for their simplicity, but we can expect that changing them to more realistic—still homogeneous—ones, would not significantly modify the qualitative aspects of our results.

This boundary value problem is invariant under the action of the group of orthogonal transformations in $\mathbf{R}^3$, which we denote by $O(3)$, defined as follows: For any $g \in O(3)$, we define the action of g by

$$(\mathbf{u}, \Theta, \mathbf{b})(x) \longrightarrow (g\mathbf{u}, \Theta, \det(g)g\mathbf{b})(g^{-1}x).$$

Note the term $\det(g)$ in the action on the $\mathbf{b}$ component, meaning that $\mathbf{b}$ is a *pseudo–vector field*.

There is an additional symmetry, which consists in changing $\mathbf{b}$ to $-\mathbf{b}$ while keeping all other variables unchanged. This symmetry, which we denote by S, will play a significant role in Section 3. We denote by G the complete symmetry group of the model, i.e., the group spanned by $O(3)$ and S, which is isomorphic to the direct product $O(3) \times Z_2$.

It is clear from these equations that the condition $\mathbf{b} = 0$ reduces the problem to the usual convection problem. Moreover, as long as $||\mathbf{u}||$ is small (in L_2 norm, say), magnetic perturbations of the convective flow are asymptotically damped. If the initial flow is steady, the *kinematic dynamo* equations (4), (5) are *autonomous linear* equations for $\mathbf{b}$. The stability problem then reduces to a spectral equation for this linear operator. Our idea is to exploit the knowledge of the primary branches of bifurcated flows (from the basic pure conduction state) in order to study numerically this spectral problem. Before we tackle this problem, we recall in the next section the known facts about the primary bifurcation of pure convective flows.

2.2. The Primary Bifurcation to Purely Convective Flow

Here we set $\mathbf{b} = 0$, so that equations (4) and (5) are automatically satisfied and the bifurcation problem reduces to the onset of convection with $O(3)$ symmetry, the transformation S $\mathbf{b} \to -\mathbf{b}$ acting of course trivially on the convective flow. A classical linear stability analysis [3] of the pure conduction state shows that there exists a critical Rayleigh number Ra_c at which the trivial solution becomes unstable to perturbations that belong to a space of spherical harmonics of a certain degree ℓ_0. This degree depends on the aspect ratio η of the spherical shell. It has been shown that ℓ_0 tends to increase to infinity as η tends to 1 ([4]). This is of course a consequence of the spherical symmetry of the problem. More precisely, for this value Ra_c, 0 is an eigenvalue of the linear operator and its eigenspace V_0 is associated with an irreducible representation of degree ℓ_o of the group $SO(3)$, i.e., with spherical harmonics $Y_m^{\ell_o}(\theta, \varphi)$, $-\ell_o < m < \ell_o$; therefore, the eigenspace is $2\ell_o + 1$-dimensional.

The standard procedures for computing bifurcated solutions (projection onto the center manifold or Lyapunov-Schmidt reduction) consist in decomposing first the unknown fields (velocity and temperature, in this case) into a leading part that belongs to V_0 and higher-order terms that sit in a complement to V_0 in the functional space where the equations are studied. This means that we decompose the velocity field as

$$\mathbf{u} = \mathbf{u}_o + \mathbf{w},$$

where $\mathbf{u}_0 \in V_0$ and $\mathbf{w}$ stands for higher-order terms that can be computed in terms of $\mathbf{u}_o$. The same decomposition holds for the temperature deviation: $\Theta = \Theta_0 + \tau$. Therefore, the problem reduces to an equation for $(\mathbf{u}_0, \Theta_0)$ in V_0, with parameter $Ra - Ra_c$. A convenient program that resolves this linear stability problem has been written by P. Laure ([17]). The second step is to compute branches of steady-states from these equations and their stability.

The velocity field $\mathbf{u}$ can be expanded along spherical harmonics by means of the so-called generalized spherical functions [9]. For $\mathbf{u}_o$ the expansion is finite because it is performed in the eigenspace of critical modes for the onset of convection, which is itself associated with the representation of degree ℓ_0 of $SO(3)$. More precisely, we can write

$$\mathbf{u}_o(r, \theta, \varphi) = \sum_{m=-\ell_0}^{m=\ell_0} \xi_m^{\ell_0}, \tag{8}$$

where $\xi_m^{\ell_0} \in V_0$ is the component of $\mathbf{u}_o$ along the direction associated with the spherical harmonic $Y_m^{\ell_o}$.

Let us set $u^o = u_r$, $u^+ = -\frac{\sqrt{2}}{2}(u_\varphi + i u_\theta)$, $u^- = \frac{\sqrt{2}}{2}(u_\varphi - i u_\theta)$. Then,

$$\xi_m^{\ell_o} = z_m \begin{bmatrix} u(r)\ T_{o,m}^{\ell_o}(\frac{\pi}{2} - \varphi, \theta) \\ v(r)\ T_{+,m}^{\ell_o}(\frac{\pi}{2} - \varphi, \theta) \\ v(r)\ T_{-,m}^{\ell_o}(\frac{\pi}{2} - \varphi, \theta) \end{bmatrix},$$

where the complex amplitudes z_m satisfy the reality condition $z_{-m} = (-1)^m \bar{z}_m$ and u and v are real functions scaled so that $\max_{\eta<r<1} u(r) = 1$.

The action of $O(3)$ on the basis elements ξ_m^ℓ is defined by its action on the radial components $T_{o,m}^\ell(\frac{\pi}{2} - \phi, \theta, 0)$ $(-\ell < m < \ell)$, which are equal to the spherical harmonics $Y_m^\ell(\theta, \phi)$ up to a normalizing constant. In particular, we have

$$T_{o,-m}^\ell = (-1)^m \bar{T}_{o,m}^\ell.$$

The action of some elements of $O(3)$ is easy to write. Let σ denote the action of antipodal symmetry (reflection through the origin in $\mathbf{R}^3$), R_ϕ the action of the rotation of angle ϕ around $0z$, and χ the action of the rotation of angle π about the axis $0x$. Then,

$$R_\phi T_{o,m}^\ell = e^{im\phi} T_{o,m}^\ell \qquad \text{and} \qquad \chi\ T_{o,m}^\ell = (-1)^{\ell+m} T_{o,-m}^\ell, \tag{9}$$

whereas the reflection σ possesses two possible representations (resp. called natural and antinatural):

$$\sigma\ T_{o,m}^\ell = (-1)^\ell T_{o,m}^\ell \qquad \text{or} \qquad \sigma\ T_{o,m}^\ell = (-1)^{\ell+1} T_{o,m}^\ell. \tag{10}$$

The velocity field is associated with the natural representation because it behaves as usual vectors do under reflection. On the contrary, the magnetic field is a *pseudovector*;

Table 1. Critical nodes and Rayleigh numbers for the cases of interest in Section 3.

Aspect Ratio η	Critical Mode ℓ_o	Critical Rayleigh Ra_c
0.1	1	102.8
0.3	2	135.2
0.4	3	166.8

hence, it should transform as the antinatural representations do. However, this system commutes with the transformation $S\ \mathbf{b} \mapsto -\mathbf{b}$. Therefore, natural components can also arise in the expansion in spherical harmonics of the solutions of this system (in [26], this distinction between natural and antinatural representations related to different classes of PDE's is discussed in the case of Euclidean two-dimensional symmetry).

Equations for the amplitudes z_m have been derived, both theoretically by taking account of the $O(3)$-invariant property of the problem (see a review in [6]), and numerically by computing the leading part of these equations (see [5] and [2]). The solutions in V_0 are selected by the nonlinearity of the problem and the numerical value of coefficients in the bifurcation equations.

We shall note E the reflection through the equatorial plane, i.e., $E = \sigma\ R_\pi$ (this is the notation in [13]).

In Table 1, we recall the critical values ℓ_0 and Ra_c for three different values of the aspect ratio η, and in the case when $g_0 = h_0 = 0$. Note that $g^*(\mathbf{r}) = h^*(\mathbf{r})$ ensures self-adjointness of the linear part of the equations (1) to (3). Such tables were first derived by Chandrasekhar [3]. We indicate here those cases in which we shall analyse the dynamo bifurcation.

Case ℓ_o = 1. All solutions are axisymmetric in this case because V_0 is isomorphic to $\mathbf{R}^3$ with the natural action of $O(3)$. More precisely, the symmetry group of a solution consists of the rotations around a certain axis and the reflections through any plane containing this axis. It is noted $O(2)^-$ in [11]. In other words, if the axis of symmetry is chosen as $0z$ (the vertical axis), then the solution is invariant under the combined actions of R_Φ and $\sigma\ .\ \chi$ (the reflection through the plane $y0z$). This primary bifurcation is a pitchfork and supercritical, any bifurcated solution being stable.

A typical bifurcated steady-state velocity that is parametrized by ϵ is expressed, in V_0, as

$$\mathbf{u}_o = \xi_o^1 = \epsilon \begin{bmatrix} u(r)\ \cos(\theta) \\ v(r)\ \sin(\theta) \\ 0 \end{bmatrix}, \tag{11}$$

$$Ra - Ra_c = O(\epsilon^2). \tag{12}$$

In other words, we have set $z_0 = \epsilon$ real in (8). The pattern of such solutions is depicted in Figure 1, which is a single convective motion cell with a vertical axis of symmetry.

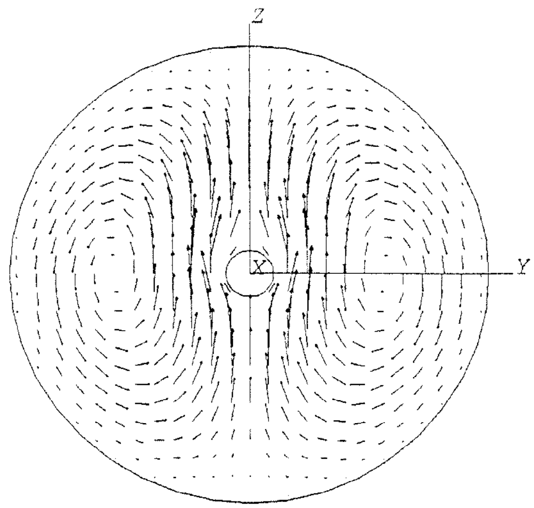


Fig. 1. $\eta = 0.1$. Velocity field in a meridian plane. The axisymetric one-cell flow is bidimensional.

Case ℓ_o = 2. Here there is in general only one branch of bifurcated solutions, which are again steady-state and axisymmetric. However, the symmetry group of these solutions is larger than in the $\ell_o = 1$ case: It contains essentially all the symmetries of a cylinder (with finite length). It is therefore generated by $O(2)$ (as a subgroup of $SO(3)$) and by σ. This branch is no longer a pitchfork branch: It is instead *transcritical*. Moreover, the bifurcated solutions are always unstable near onset [1]. The corresponding solutions (with vertical axis of symmetry) read

$$\mathbf{u}_o = \xi_o^2 \; = \; \epsilon \begin{bmatrix} \frac{1}{2}\, u \; (-1 + 3\cos(\theta)^2) \\ \sqrt{3}\, v \;\; \sin(\theta)\cos(\theta) \\ 0 \end{bmatrix}, \tag{13}$$

$$R_a - R_{ac} \; = \; O(\epsilon). \tag{14}$$

The pattern produced by this flow is shown in Figure 2.

It turns out that if $g_*(r)/h_*(r)$ is nearly constant (see Section 2.1), a degeneracy is introduced into the bifurcation equations that allows for a *turning point* on the subcritical part of the branch *near the bifurcation point* ([4] and [11]). It follows that there are *two types* of axisymmetric solutions coexisting supercritically: The main difference between the two is that the flow is going outwards at the poles and inward at the equator for one type, and the opposite way for the other type (corresponding to ϵ being positive or negative in the above expression for the flow). However, these two solutions are not related by any symmetry. It was found in [5] that when the bifurcation is exactly

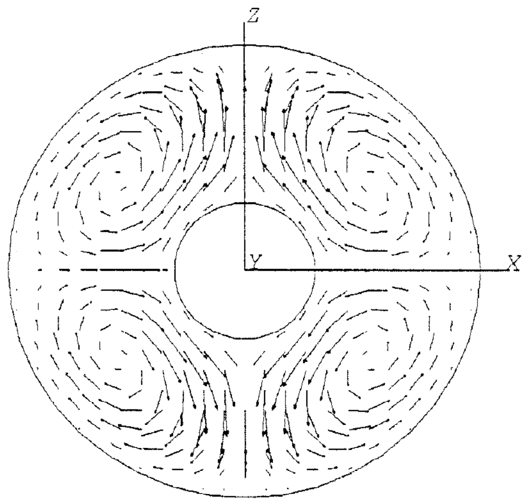


Fig. 2. $\eta = 0.3$. Velocity field in a meridian plane. The flow is axisymmetric bidimensional and admits equatorial symmetry. It therefore consists of two symmetric cells surrounding the polar axis. This flow is upwelling at the poles and downwelling at the equator. Solutions with reverse flow are possible too, but are unstable.

supercritical (this happens when $g_*(r) = h_*(r)$), the stable branch corresponds to the upwelling convection near poles and downwelling at the equator with ϵ positive.

Another consequence of this degeneracy is that a secondary branch of steady-states can join the two branches of axisymmetric solutions (see [11] and [5]). These secondary solutions are not axisymmetric anymore: They have the symmetry of a parallelepiped, i.e., the symmetry group is generated by D_2 (itself generated by R_π and χ) and by σ. They have been observed in direct numerical simulations by Young [28].

The space V_0 in the case $\ell_0 = 2$ has some special features that are important to describe here. First, any element in V_0 is fixed under the reflection σ (natural representation with ℓ even, see (10)). Second, it is spanned by letting $SO(3)$ act on a flow-invariant plane P. This plane is the subspace of those elements that are fixed by (at least) the action of the group D_2 (we can forget, in this case, about the reflections, because every element in V_0 is fixed by σ). Of course, this implies that every element in V_0 belongs to a certain flow-invariant plane that is a copy of P under a certain rotation. Therefore, the bifurcation and stability problem reduces to what happens in P. This plane can be chosen so that it is spanned by ξ_0^2 and the real part of ξ_2^2. It contains three invariant axes, which are mapped one into another by a rotation of angle $2\pi/3$ in P, and which correspond to axisymmetric elements with axis of symmetry mapped to each other by a certain rotation in $\mathbf{R}^3$. More precisely, the axis generated by ξ_0^2 corresponds to axisymmetric flows with vertical axis of symmetry (i.e., the z-axis), and the two other axes correspond to flows with horizontal axis of symmetry (namely, the x and y-axes).

We can represent a general flow belonging to P by parametrising the axial direction in this plane with an angle γ:

$$\mathbf{u}_o = \epsilon \begin{bmatrix} \frac{u}{2} \left(\cos(\gamma)\ (3\cos(\theta)^2 - 1) + \sqrt{3}\sin(\gamma)\ (1 - \cos(\theta)^2)\cos(2\varphi)\right) \\ v\ \sin(\theta)\cos(\theta)\left(\sqrt{3}\ \cos(\gamma) - \sin(\gamma)\cos(2\varphi)\right) \\ v\ \sin(\gamma)\ \sin(\theta)\ \sin(2\varphi) \end{bmatrix}. \quad (15)$$

The three invariant axes are then parametrised by taking $\gamma = 0$, $2\pi/3$, and $\pi/3$, respectively. The parallelepipedic pattern displayed by such a velocity field is similar, in any meridian plane, to Figure 2. The main difference is due to a nonzero periodic azimuthal velocity u_φ.

Case $\ell_o = 3$. This case was first treated by Busse and Riahi [2], who showed the existence of three different types of steady-state solutions and described their stability. It was shown by [6] that no other solution can bifurcate in general. One of the branches consists of axisymmetric flows, but it was proven that these solutions are always unstable although the bifurcation is of pitchfork type. The stable solutions can have either full tetrahedral symmetry or hexagonal-like symmetry. The symmetry group of the former solution is generated by the group T of direct symmetries of a tetrahedron (a tetrahedron has four threefold axes and three twofold axes) and by the reflection through the plane perpendicular to a threefold rotation axis in T. It was denoted by O^- in [11] because it is isomorphic to the octahedral group O. It can also be seen as the group generated by the reflections through the six planes $x = \pm y$, $y = \pm z$, and $x = \pm z$ in $\mathbf{R}^3$. In this case, the twofold rotation axes are the coordinate axes, the threefold axes are defined by $x = \pm y = \pm z$, and the six planes of reflection are those spanned by one twofold axis and one threefold axis.

For the tetrahedral solution, the bifurcated velocity whose axis orientation is the same as described above, has the form

$$\mathbf{u}_o = \xi^3_{-2} + \xi^3_2 \ = \ \epsilon \begin{bmatrix} -\frac{\sqrt{15}}{\sqrt{2}}\ u\ \cos(\theta)\ (\cos(\theta)^2 - 1)\ \sin(2\varphi) \\ \frac{\sqrt{5}}{2}\ v\ \sin(\theta)\ (3\ \cos(\theta)^2 - 1)\ \sin(2\varphi) \\ \sqrt{5}\ v\ \cos(\theta)\ \cos(2\varphi) \end{bmatrix}, \quad (16)$$

$$R_a - R_{ac} \ = \ O(\epsilon^2). \quad (17)$$

The symmetry group of the other solution is generated by the antirotation $\sigma\ R_{\frac{\pi}{3}}$ (we recall that σ is the reflection through the origin and $R_{\frac{\pi}{3}}$ is the sixfold rotation around the vertical axis) and by a rotation of angle π around a horizontal axis (e.g., χ). This group is isomorphic to the dihedral group D_6 and is denoted by D^d_6 in [11]. This is why we call this symmetry “hexagonal-like.”

For the hexagonal-like solution, we have

$$\mathbf{u}_o = \xi^3_{-3} + \xi^3_3 \ = \ \epsilon \begin{bmatrix} -\frac{\sqrt{5}}{2}\ u\ \sin(\theta)\ (\cos(\theta)^2 - 1)\ \sin(3\varphi) \\ \frac{\sqrt{15}}{2\sqrt{2}}\ v\ \cos(\theta)\ (\cos(\theta)^2 - 1)\ \sin(3\varphi) \\ \frac{\sqrt{15}}{2\sqrt{2}}\ v\ (\cos(\theta)^2 - 1)\ \cos(3\varphi) \end{bmatrix}, \quad (18)$$

$$R_a - R_{ac} \ = \ O(\epsilon^2). \quad (19)$$

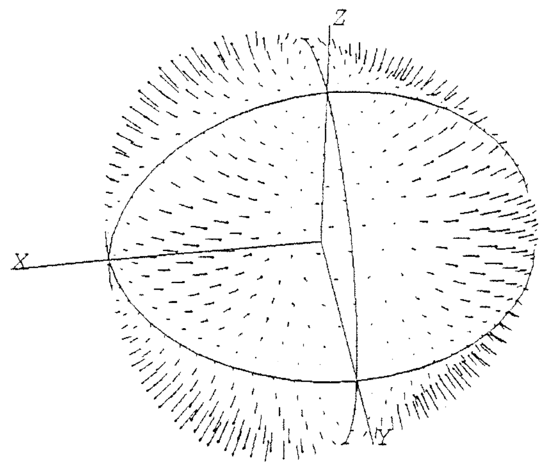


Fig. 3. $\eta = 0.4$. Velocity field with tetrahedral symmetry at fixed radius $r \approx 0.6$. The pattern on the sphere looks octahedral, but the velocity in any two adjacent "cells" points inward in one cell and outward in the other.

Notice that the above vector field is invariant by rotation of angle π around the y-axis, i.e., $R_\pi.\chi$, instead of χ.

The tetrahedral and hexagonal-like patterns are shown in Figures 3 and 4 respectively.

3. The Dynamo Instability

The question we address now is the following: Given a convective motion $\mathbf{u}$ of the fluid, does there exist a critical value of its energy (i.e., of $\|\mathbf{u}\|$ in a suitable L^2 norm) at which a dynamo can develop? From the form of the model equations, we see that the critical stability condition for a dynamo action depends on the spectrum of the linear equations (4), (5) ($\mathbf{u}$ being imposed). If $\|\mathbf{u}\|$ is small enough, a perturbation argument shows that the spectrum consists of isolated eigenvalues with finite multiplicity and negative real parts. So the question can be made more precise: Does this spectrum cross the imaginary axis when $\|\mathbf{u}\|$ is increased, and when this happens, what type of bifurcation does occur?

Let us first make some general comments about this bifurcation problem. We are looking for a *secondary bifurcation* from a steady-state solution (convective flow) that has already broken the initial spherical symmetry of the model equations. Hence, this primary solution is not isolated, but instead belongs to an $SO(3)$ group orbit, which is a connected set in the G orbit of steady-states. Note that when such a steady-state is said to be stable, it is meant that its G orbit is attracting. The directions along the G orbit are neutrally stable. This group orbit is two-dimensional when the flow is axisymmetric and three-dimensional otherwise (its dimension is given by the dimension of the Lie

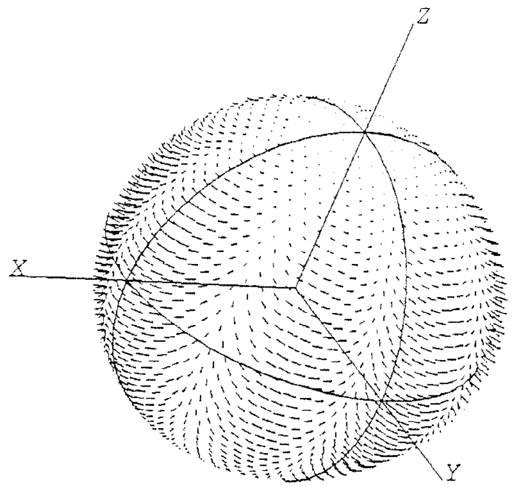


Fig. 4. $\eta = 0.4$. Velocity field with hexagonal-like symmetry at fixed radius $r \approx 0.6$. Here the sphere is divided into six meridian "cells," themselves divided into two parts by the equatorial plane.

group $SO(3)$ minus the dimension of the subgroup of those rotations which keep the solution invariant). As a consequence, the linear operator at this solution always has a zero eigenvalue: Indeed, its kernel contains the tangent space to the group orbit at that solution. Formally, the proof of this is easy: Let $T(g)$ denote the action of $O(3)$ in the Hilbert space $\mathcal{H}$ in which the evolution equation is defined. Let $Z = (\mathbf{u}, \Theta, \mathbf{b})$ and Z_0 denote a steady-state solution of the evolution equation. Then,

$$\frac{dZ}{dt} = F(Z, \delta).$$

Let $g(\varepsilon)$ be any smooth curve in $O(3)$ such that $g(0) = Id$. Since F commutes with $T(g)$, we can write $F(T(g(\varepsilon))Z_0, \delta) = 0$ for all ε; hence, by differentiation at $\varepsilon = 0$, we get

$$D_z F(Z_0, \delta) J Z_0 = 0,$$

where $J = \frac{dT(g)}{d\varepsilon}\ |_{\varepsilon=0}$. It can be justified that J is well-defined as the infinitesimal generator of the strongly continuous group of transformations T, and JZ_0 belongs to the tangent space in $\mathcal{H}$ to the group orbit at Z_0.

It follows that the usual bifurcation theory needs to be modified in order to take these "neutral modes" into account. This is accomplished by locally projecting the equations onto the normal section to the group orbit in order to eliminate the parasitic zero eigenvalues in the center manifold reduction. This procedure was applied in the simpler context of an $O(2)$ group action (Couette-Taylor problem) by [14]. Moreover, it is known that the equations projected on the tangent space to the group orbit may have a nontrivial component, which results in a slow, uniform drift along this group orbit (for the bifurcated solutions). The drifting solutions can be seen as *rotating waves* with a nonzero but small frequency. There is a purely algebraic condition for the existence of such a drift, as shown by [8] and [16]. Let Σ be the isotropy group of the bifurcated solution (i.e., the subgroup of all transformations keeping this solution invariant) and $N(\Sigma)$ denote the normalizer of Σ in G. Then,

Theorem 1. *The bifurcated solution from the group orbit of Z_0 has the form* $\exp(\omega J t)\tilde{Z}$, *where $\tilde{Z}$ is independent of time, J is an infinitesimal generator for the action of $SO(3)$, and ω is either 0 if $\dim N(\Sigma)/\Sigma = 0$, or it is generically of order r^k for some integer $k > 0$, r being the size of the solution, if $\dim N(\Sigma)/\Sigma = 1$. In the latter case, the solution is a rotating wave.*

The following lemma will help us to determine the normalizer of $\Sigma \subset G$ (recall that $G = O(3) \times Z_2$, where Z_2 is the two-element group generated by the symmetry S). We denote by π the projection $G \to O(3)$.

Lemma 1. $N(\Sigma) = N_1 \times Z_2$, *where*

$$N_1 = N_{O(3)}(\pi(\Sigma)) \cap N_{O(3)}(\Sigma \cap O(3)).$$

The proof of this lemma is identical to the proof of lemma A2 in [6].

The situation in our problem is simplified by the fact that the linear magnetic disturbances automatically belong to the normal space to the group orbit at $\mathbf{u}$. To be more precise, the space $\mathcal{H}$ can be written $\mathcal{H}_{u,\theta} \times \mathcal{H}_b$, where $\mathcal{H}_{u,\theta}$ is the space of $(\mathbf{u}, \Theta)$ components and $\mathcal{H}_b$ is the space of $\mathbf{b}$ components. Since the group orbit of $\mathbf{u}$ lies in $\mathcal{H}_{u,\theta}$, the space $\mathcal{H}_b$ is normal to it. Hence, solving the kinematic dynamo equations (or the related spectral problem) directly leads to the relevant critical modes. From the knowledge of the possible groups Σ associated with bifurcated solutions, it will therefore be possible not only to predict the possible patterns for these solutions, but also to tell whether a uniform drift is to be expected.

We now turn to the spectral problem and look for solutions of the form $e^{\lambda t}\mathbf{b}$, $\mathbf{b}$ independent of t, of equations (4) and (5). The kinematic dynamo equation (4) then reads

$$\lambda \mathbf{b} = \beta \triangle \mathbf{b} + \nabla \times (\epsilon \bar{\mathbf{u}}_0 \times \mathbf{b}) + O(\epsilon^2),$$

where $\bar{\mathbf{u}}_o$ is one of the primary solutions described in the previous section, normalized in such a way that

$$\bar{\mathbf{u}}^o = \sum_{m=-\ell_o}^{m=\ell_o} z_m^{\ell_o} \begin{bmatrix} u\, T_{o,m}^{\ell_o} \\ v\, T_{+,m}^{\ell_o} \\ v\, T_{-,m}^{\ell_o} \end{bmatrix} \qquad \text{and} \qquad \sqrt{\sum_{m=-\ell_o}^{m=\ell_o} |z_m^{\ell_o}|^2} = 1. \tag{20}$$

If both ϵ^2/β and ϵ are small, then classical perturbation theory of linear operators shows that the eigenvalues λ will have the same leading part as those of $\beta \mathcal{J}_\delta$, where $\delta = \epsilon/\beta$ and

$$\mathcal{J}_\delta \mathbf{b} = \triangle \mathbf{b} + \delta\, \nabla \times\ (\bar{\mathbf{u}}_o \times \mathbf{b}), \tag{21}$$

with the divergence-free condition $\nabla.\mathbf{b} = 0$.

The method used to solve this eigenvalue problem relies on the decomposition of (21) on an adapted basis. As we did in Section 2.2 for **u**, we set

$$b^o = b_r, \qquad b^+ = -\frac{\sqrt{2}}{2}(b_\varphi + i b_\theta), \qquad b^- = \frac{\sqrt{2}}{2}(b_\varphi - i b_\theta).$$

Writing $\mathbf{b} = (b^o, b^+, b^-)$,we can expand each component b^n, $n = 0, +1, -1$, in terms of the so-called generalised spherical functions (see Appendix A):

$$b^n = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{m=\ell} b^\ell_{n,m}(r)\ T^\ell_{n,m}(\frac{\pi}{2} - \varphi, \theta), \tag{22}$$

with the reality condition $b^\ell_{n,-m} = (-1)^m \overline{b}^\ell_{-n,m}$.

Observe that for $\delta = 0$, we have $\mathcal{J}_0 \mathbf{b} = \triangle \mathbf{b}$, which is invariant under the action of $O(3)$. Hence, in this case, the radial functions $b^\ell_{n,m}(r)$ are in fact independent of m.

After the expansions (8), (22) have been introduced into (21) and this equation has been projected onto the space of generalised spherical functions, a set of ordinary differential equations in r is obtained for the variables $b^o(r), b^+(r), b^-(r)$. The details for the derivation of the resulting system of equations are carried forward to the Appendix, where it is shown how we can write the algebraic eigenvalue problem in the matrix form,

$$M(\delta)\mathbf{b} \ = \ \lambda\ B\ \mathbf{b},$$

where M is a sparse matrix and B a noninvertible diagonal matrix. This generalised eigenvalue problem is solved by an Arnoldi method mixed with an inverse iterative method. This method is well adapted to determine the eigenvalues and the corresponding eigenvectors of large unsymmetric matrices [20]. This technique is based upon a reduction to an upper-Hessenberg matrix of lower dimension by a Gram-Schmidt othogonalisation process starting from an arbitrary vector [23]. The discretization scheme in the radial direction is a finite difference with N discretization points. The range for the spherical harmonics has been truncated at $\ell = L$. The efficiency of this method has been extensively tested for the $\ell_0 = 2$ axisymmetric convective flow. The influence of radial discretization N and the spherical harmonic truncation L are shown in Tables 2 and 3. The critical δ is always reached for a zero eigenvalue and the bifurcated solution will be stationary. In this way, we show that the magnetic instability is again detected if we go further in our truncation. Consequently, the critical mode is well described by few spherical harmonics, and we have fixed for the sequel $N = 40$ and $L = 7$.

Here it must be noted that the critical magnetic Reynolds numbers that we have computed do not depend on the number of discretization points, nor do they depend on the number of spherical harmonics. The convective flows that we have selected from the primary bifurcation analysis seem to make fewer numerical problems than those that have been "artificially" built in previous works [12].

One of our main tasks being to determine the patterns of the bifurcated solutions, we need to determine the *type* of symmetry-breaking bifurcation that results from the appearance of a dynamo instability. Let G_0 denote the symmetry group (or isotropy group) of the purely convective flow under consideration. Note that $S \in G_0$ in any case. Suppose, to simplify, that the critical eigenvalues of $\mathcal{J}_{\delta_c}$ are essentially concentrated at

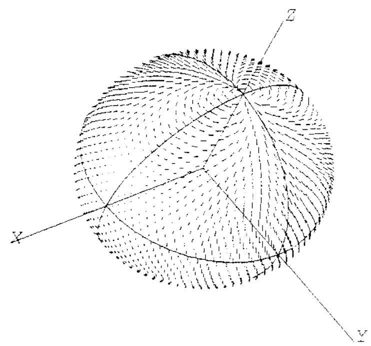
Table 2. Influence of radial discretisation N on the critical value of δ ($L = 7$).

N	Size of M	Number of Nonzero Terms	δ_c
10	1 890	30 330	12.84
20	3 780	30 330	13.24
30	5 670	46 620	13.25
40	7 560	62 910	13.25
50	9 450	79 200	13.25
60	11 340	95 490	13.25

Table 3. Influence of mode truncation L on the critical value of δ ($N = 40$).

L	Size of M	Number of Nonzero Terms	δ_c
3	1 800	10 766	13.75
4	2 880	19 764	13.39
5	4 200	31 454	13.28
6	5 760	45 836	13.27
7	7 560	62 910	13.25
8	9 600	82 676	13.25
9	11 880	105 134	13.25
10	14 400	130 284	13.25

zero (which will in fact be the case in the examples we have studied). The space of critical modes is $V_0 = Ker(\mathcal{J}_{\delta_c})$, and because $\mathcal{J}_{\delta_c}$ commutes with the action of G_0 by construction, V_0 is G_0-invariant. We are therefore faced with a bifurcation problem with G_0 symmetry, and the question is now to determine the action (representation) of G_0 in V_0. Because only one parameter δ is involved in this problem, it is known that typically, the space V_0 is associated with an *absolutely irreducible representation* of G_0 (see [11]). This problem was tackled in the planar case ([25]) by decomposing the operator $\mathcal{J}_\delta$ along the *isotypic components* of the function space with respect to the action of G_0, and following the eigenvalues in each isotypic component from the parameter value $\delta = 0$. The isotypic decomposition of a group representation consists in grouping together its equivalent irreducible representations. Then the representation space $\mathcal{H}$ (which is a Hilbert space in our case) decomposes in a unique way into a direct sum of subspaces that are associated with these classes of irreducible representations: $\mathcal{H} = \oplus\mathcal{H}^{(\mu)}$. Moreover, any linear operator that commutes with this action admits a unique decomposition along the $\mathcal{H}^{(\mu)}$'s (see, e.g., [18]). Therefore, if one knows this isotypic decomposition, it is possible to restrict the spectral analysis to the isotypic components, allowing one first to simplify the numerical calculation and second to identify easily the type of irreducible representation associated with the critical eigenvalue. In the spherical case, unfortunately, it is not so easy to identify the isotypic components of the group G_0, especially when this group is an exceptional subgroup of $O(3)$ (like the group O^- described in the previous



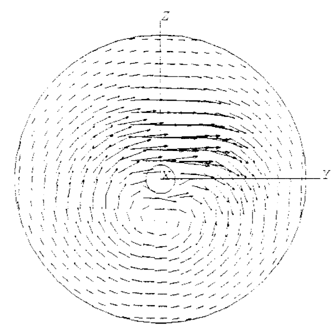


Fig. 5. $\eta = 0.1$. Bifurcated magnetic field: (a) at constant $r \approx 0.5$, (b) on the meridian plane $x = 0$. Note the concentration of energy near the north pole of the inner sphere and the vanishing energy near the south pole of the inner sphere.

Section 2.2 in the case $\ell_0 = 3$). We have therefore chosen to solve the eigenvalue problem "globally," by the above-mentioned numerical method, and then to determine the representation of G_0 by simple examination of the critical eigenvectors. This proved to work well in all cases we have studied. Of course, we can examine the action of G_0 only on a finite number of modes in the harmonic expansion of the eigenvectors. However, as long as no "big" degeneracy has been found at low order and by a genericity argument about the irreducible character of the action, we can conclude with a high degree of confidence that the action so obtained is the actual action of G_0 on the critical eigenspace.

We now turn to the case study.

3.1. Case 1. $\eta = 0.1$*: Axisymmetric Primary Flow*

The convective solution for $\ell_o = 1$ is written in (13) for a flow that is axisymmetric around the z-axis. Surprisingly enough, for such a "simple" velocity field we have found a dynamo action at the relatively low value $\delta_c \approx 9$. The critical components are all associated with the value $m = 1$, i.e.,

$$\mathbf{b}_c = \sum_{\ell=1}^{\infty} \left(\xi_{-1}^{\ell} + \xi_1^{\ell}\right) \qquad \text{with } \xi_1^{l} = \begin{bmatrix} b_o^{\ell}(r)\ T_{o,1}^{\ell}(\pi/2 - \varphi, \theta) \\ b_+^{\ell}(r)\ T_{+,1}^{\ell}(\pi/2 - \varphi, \theta) \\ b_-^{\ell}(r)\ T_{-,1}^{\ell}(\pi/2 - \varphi, \theta) \end{bmatrix}, \tag{23}$$

and b_o^{ℓ}, b_+^{ℓ} and b_-^{ℓ} real. In spherical coordinates,

$$\mathbf{b}_c = \begin{bmatrix} f_r(r,\theta)\ \sin(\varphi) \\ f_\theta(r,\theta)\ \sin(\varphi) \\ f_\varphi(r,\theta)\ \cos(\varphi) \end{bmatrix}. \tag{24}$$

This field is plotted in Figure 5.

Symmetry-breaking bifurcation. The symmetry group G_0 is generated by $O(2)^-$ (see Section 2.2, case $\ell_0 = 1$) and S. The critical eigenvalue is zero and double. The reason for multiplicity 2 is that the rotational symmetry $SO(2)$ is broken by the critical eigenvectors (action on φ). It can be checked that the reflection through the "vertical" plane yOz keeps $\mathbf{b}_c$ fixed. Hence, this bifurcation is equivalent to the usual bifurcation with $O(2)$ symmetry, the action of $O(2)^-$ on the critical eigenmodes being then equivalent to the action of $O(2)$ in the plane. It follows that the bifurcated branch is of pitchfork type.

We must also take the symmetry S (i.e., $\mathbf{b} \to -\mathbf{b}$) into account. The combined action of R_π (rotation by π around the vertical axis; see (9)) and S keeps $\mathbf{b}_c$ fixed. It follows that the full symmetry group Σ of a bifurcated solution is spanned by the following elements: (1) the reflection through a vertical plane, (2) $S.R_\pi$.

Thanks to Lemma 1 and using the list of normalizers of subgroups of $O(3)$ in [6], it is straightforward to check that $N(\Sigma) = N_1 \times Z_2$, where N_1 is spanned by D_4 (subgroup of $SO(3)$) and by the antipodal symmetry σ. Hence, $\dim N(\Sigma)/\Sigma$ is finite, which implies, thanks to Theorem 3.1, that the bifurcated solutions are genuine equilibria (no drifting frequency along the G group orbits).

Note the role played by the symmetry S here. If it were not present, Σ would be reduced to the two element group spanned by the reflection through a vertical plane, the normalizer of which contains $SO(2)$. Then a drift would, in general, be present.

3.2. Case 2. η = 0.3: Axisymmetric Primary Flow

We have seen that in this case, $\ell_o = 2$ and the axisymmetric solutions to the primary bifurcation problem are invariant under the actions of $O(2)$ (as a subgroup of $SO(3)$) and of σ (and of course of S).

As noted in section 2, there are two possible solutions depending on whether the convective flow is upward or downward at the poles. Only the state with upward convection at the poles is stable in the self-adjoint case. However, depending on the kind of deviation to self-adjointness, both solutions can gain stability when this limiting situation is perturbed. So, we have to do the stability analysis for the two velocity fields,

$$\bar{\mathbf{u}}^o = \xi_0^2 \qquad \text{or} \qquad -\xi_0^2.$$

The variation of the most unstable eigenvalue with δ is plotted in Figure 6 for both solutions, and we see that it is the stable flow, upwelling at the poles, that is destabilized first by a dynamo mechanism.

The critical parameter value δ_c is ≈ 13.25. Here again the critical eigenvalue is found to be double, a critical mode reading

$$\mathbf{b}_c = \sum_{\ell=1}^{\infty} \left(\xi_{-1}^{\ell} + \xi_1^{\ell}\right), \tag{25}$$

with $\xi_1^l = \begin{bmatrix} b_o^\ell(r)\ T_{o,1}^\ell \\ b_+^\ell(r)\ T_{+,1}^\ell \\ b_-^\ell(r)\ T_{-,1}^\ell \end{bmatrix}$ or $\xi_1^l = \begin{bmatrix} 0 \\ b_+^\ell(r)\ T_{+,1}^\ell \\ -\,b_-^\ell(r)\ T_{-,1}^\ell \end{bmatrix}$ for odd and even ℓ, respectively.

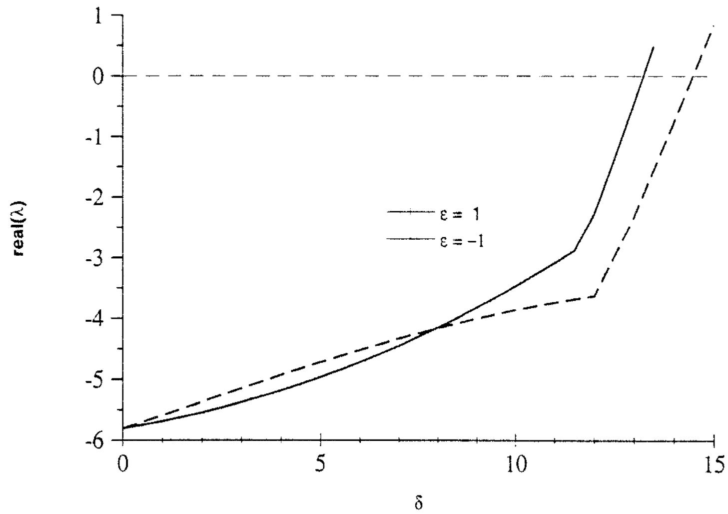


Fig. 6. Real part of the most unstable eigenvalue versus δ for the two axisymmetric solutions in the case $\ell_o = 2$. ϵ is defined by the relation (13).

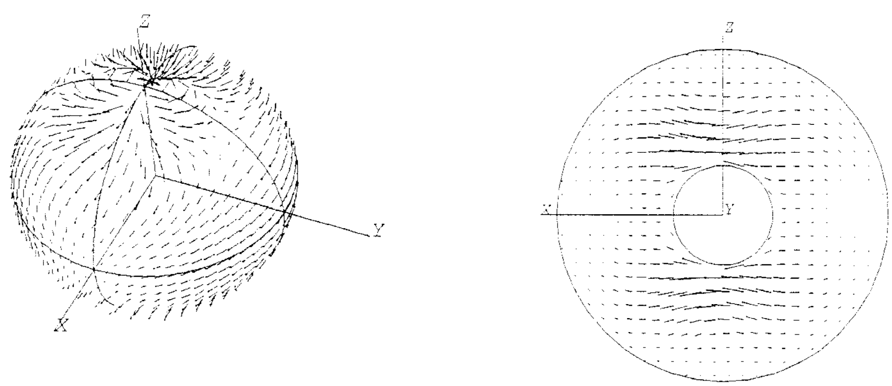


Fig. 7. $\eta_o = 0.3$. Bifurcated magnetic field in the case of an axisymmetric basic flow. (a) At constant $r \approx 0.5$, (b) on the meridian plane $y = 0$. This solution has equatorial symmetry. Energy is concentrated near the polar axis of the basic flow.

The spherical components of $\mathbf{b}_c$ have the following general expression,

$$\mathbf{b}_c = \begin{bmatrix} f_r(r,\theta)\ \cos(\varphi) \\ f_\theta(r,\theta)\ \cos(\varphi) \\ f_\varphi(r,\theta)\ \sin(\varphi) \end{bmatrix}, \tag{26}$$

and are plotted in Figure 7.

The symmetry-breaking bifurcation. As in the previous case, the $SO(2)$ symmetry is broken. On the other hand, the rotation χ around the horizontal axis $0x$ keeps $\mathbf{b}_c$ fixed. Therefore this bifurcation is the usual $O(2)$ symmetry-breaking one, i.e., is of pitchfork type. Note also that the antipodal symmetry σ is broken: $\sigma\mathbf{b}_c = -\mathbf{b}_c$. It follows that both $\sigma.R_\pi$ (reflection through the equatorial plane) and $S.\sigma$ keep $\mathbf{b}_c$ fixed. Therefore Σ is now spanned by $\chi, \sigma.R_\pi$, and $S.\sigma$. Its normalizer in G is again finite. It follows that the bifurcated solutions are pure equilibria. Note however that this situation is qualitatively different from the previous one ($\ell_0 = 1$), although the basic flow is axisymmetric in both cases. First, the present solutions are invariant by equatorial plane reflection, while the previous ones are changed to the opposite. Second, the S symmetry was crucial in the previous case to ensure that the bifurcated solutions are pure equilibria. Indeed, if this symmetry was not taken into account, the normalizer $N(\Sigma)$ would contain $SO(2)$; hence, the solutions would have a drifting (low) frequency. In the present case, this property is forced by the $O(3)$ symmetry. If one would forget about S, Σ would be the group noted D_2^z in [11] (spanned by χ and $\sigma.R_\pi$), whose normalizer in $O(3)$ is also finite (see [6]).

3.3. Case 3. $\eta = 0.3$: Nonaxisymmetric Primary Flow

We take expression (15) as an input for this velocity field, for various values of the angular coordinate γ in the range 0 to $2\pi/3$.

As shown in Figure 8, we find the same critical δ_c at $\gamma = 0$ and $2\pi/3$. This was expected because, as we indicated in Section 2.2, these values correspond to the same axisymmetric solution (with $\varepsilon > 0$), just rotated by certain angles. The peak at $\gamma = \pi/3$ also corresponds to an axisymmetric basic flow, but with $\varepsilon < 0$. Note that as long as the nonaxisymmetric convective flow exists with a γ smaller than a certain value (approximately equal to $\pi/6$), the corresponding δ_c is smaller than the one for the axisymmetric basic flow. The minimum critical δ (~ 13) is reached at $\gamma = .3$ and 1.8.

The symmetry group G_0 is generated by D_2 and σ (see Section 2.2, case $\ell_0 = 2$), and by S. This group has only one-dimensional irreducible representations; hence, we expect zero to be a simple eigenvalue (except of course for $\gamma = 0,\ \pi/3,\ 2\pi/3$). This is checked by examination of the harmonic expansion of $\mathbf{b}_c$. In fact, Σ is the same as in the case of an axisymmetric primary flow; hence, the bifurcation is a pitchfork and the solutions are again pure equilibria.

3.4. Case 4. $\eta = 0.4$: Nonaxisymmetric Primary Flows

As we pointed out in Section 2.2, there are two possible branches of stable convective solutions, which have symmetry groups denoted as D_6^d (hexagonal-like symmetry) and O^- (full tetrahedral symmetry), respectively (see Figure 9).

3.4.1. Primary Flow with Hexagonal-like Symmetry. In this case, the instability occurs at critical value $\delta_c \sim 17.5$ with the following critical eigenvector for the magnetic field:

$$\mathbf{b}_c = \sum_{\ell \text{ odd}}^{\infty} \xi_0^\ell + \sum_{\ell=1}^{\infty} \sum_{m=1}^{l/3} \xi_{\pm 3m}^\ell, \tag{27}$$

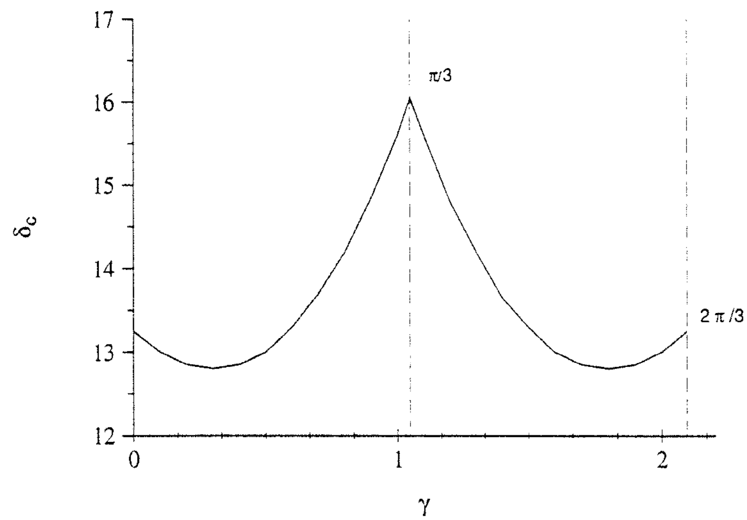


Fig. 8. Critical δ_c versus the angle γ in the case $\ell_o = 2$ ($\eta = 0.3$). γ is defined by the relation (15).

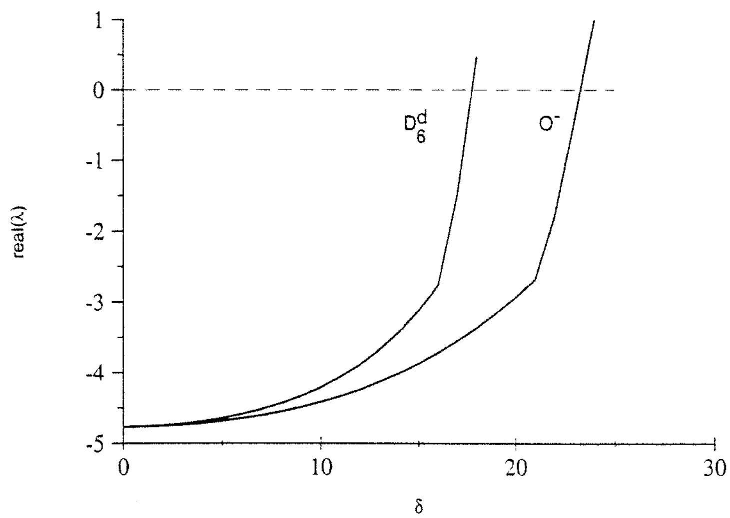


Fig. 9. Real part of the most unstable eigenvalue versus δ for the octahedral-like and hexagonal-like basic flows in the case $\eta = 0.4$.

where $b^{\ell}_{n,\pm 3m}$ are purely imaginary; $b^{\ell}_{o,3m} = 0, b^{\ell}_{-,\pm 3m} = -b^{\ell}_{+,\pm 3m}$ for ℓ odd and $b^{\ell}_{-,\pm 3m} = b^{\ell}_{+,\pm 3m}$ for ℓ even.

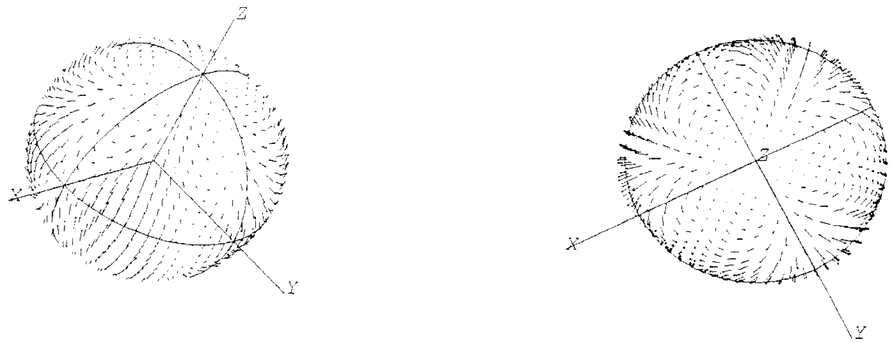


Fig. 10. $\eta = 0.4$, hexagonal-like basic flow. Bifurcated magnetic field: (a) on the sphere $r \sim 0.6$; (b) top view (note the remaining threefold symmetry).

In spherical coordinates, this reads

$$\mathbf{b}_c = \begin{bmatrix} f_r(r,\theta) \ + \ \sum_{m=1} g_{m,r}(r,\theta)\sin(3\ m\ \varphi) \\ f_\theta(r,\theta) \ + \ \sum_{m=1} g_{m,\theta}(r,\theta)\sin(3\ m\ \varphi) \\ \sum_{m=1} g_{m,\varphi}(r,\theta)\cos(3\ m\ \varphi) \end{bmatrix}. \tag{28}$$

The symmetry-breaking bifurcation. The primary flow is invariant by $\sigma.R_{\frac{\pi}{3}}$ and $R_\pi.\chi$ (see Section 2.2). Hence, G_0 is spanned by these transformations and by S. We immediately see from the above expressions that the critical eigenvectors are still invariant under the pure rotational part of the group, i.e., the rotation $R_{2\pi/3}$. This implies that the critical eigenvalue should be *simple* in this case. We find that $\sigma.R_{\frac{\pi}{3}}\ \mathbf{b}_c = -\mathbf{b}_c$. Similarly, $R_\pi.\chi.\mathbf{b}_c = -\mathbf{b}_c$. The bifurcation is clearly a pitchfork. The group Σ is generated by $S.\sigma.R_{\frac{\pi}{3}}$ and $S.R_\pi.\chi$. Its normalizer in G is again a finite group by Lemma 1 and [6]. Hence, we have again a bifurcation to pure equilibria. Figure 10 shows the corresponding pattern.

3.4.2. Primary Flow with Tetrahedral Symmetry. The bifurcation occurs in this case at $\delta_c \sim 23.32$ and the computed critical eigenvector has the form

$$\mathbf{b}_c = \sum_{\ell \text{ even}}^{\infty} \xi_0^\ell + \sum_{\ell=1}^{\infty}\sum_{m=1}^{l/2} \xi_{\pm 2m}^\ell, \tag{29}$$

in which the functions $b_{n,2\ m}^\ell$ have the following properties:

1. $b_{o,2m}^\ell$ is real if ℓ is even and imaginary if ℓ is odd;
2. $Re(b_{-,2m}^\ell) = Re(b_{+,2m}^\ell)$ and $Im(b_{-,2m}^\ell) = -Im(b_{+,2m}^\ell)$ for ℓ even;
3. $Re(b_{-,2m}^\ell) = -Re(b_{+,2m}^\ell)$ and $Im(b_{-,2m}^\ell) = Im(b_{+,2m}^\ell)$ for ℓ odd.

In spherical coordinates,

$$\mathbf{b}_c = \begin{bmatrix} f_r(r,\theta) \ + \ \sum_{m=1}(g'_{m,r}(r,\theta)\cos(2\ m\ \varphi) + g"_{m,r}(r,\theta)\sin(2\ m\ \varphi)) \\ f_\theta(r,\theta) \ + \ \sum_{m=1}(g'_{m,\theta}(r,\theta)\cos(2\ m\ \varphi) + g"_{m,\theta}(r,\theta)\sin(2\ m\ \varphi)) \\ \sum_{m=1}(g'_{m,\varphi}(r,\theta)\sin(2\ m\ \varphi) + g"_{m,\varphi}(r,\theta)\cos(2\ m\ \varphi)) \end{bmatrix}, \quad (30)$$

where the various functions g', $g"$ are trigonometric series in θ.

The symmetry-breaking bifurcation. We have seen in Section 2.2, case $\ell_0 = 3$, that the group O^- is the full symmetry group of a regular tetrahedron. The chosen input velocity field has twofold axes oriented like the usual coordinate axes in $\mathbf{R}^3$, i.e., the corresponding rotations are R_π, χ, and $R_\pi.\chi$. Inspection shows that these three rotations keep $\mathbf{b}_c$ fixed. Let us denote by D_2 the group generated by these rotations.

We expect $\mathbf{b}_c$ to belong to an irreducible representation of O^-. This group is isomorphic to the octahedral group O (the 24 direct symmetries of the cube); therefore it has the same representations: two one-dimensional irreducible representations, one two-dimensional irreducible representation and two three-dimensional irreducible representations (see, e.g., [18]). The question is to determine which of these does act on the critical eigenspace.

However, D_2 keeps $\mathbf{b}_c$ fixed and moreover D_2 is normal in O^-. Therefore, this irreducible representation must be isomorphic to an irreducible representation of the group O^-/D_2. But this group is itself isomorphic to the six-elements group D_3, which has only one- or two-dimensional irreducible representations. Since the action of O^- clearly doesn't keep the direction of $\mathbf{b}_c$ invariant, we conclude that this irreducible representation is the two-dimensional one.

Let us now consider the additional symmetry S. Its combination with $D_3 \simeq O^-/D_2$ is isomorphic to the symmetry group of hexagons D_6. It is enough to notice that S can be identified with the rotation by π in the plane. Therefore, its two-dimensional irreducible representation has the same properties as the hexagonal irreducible representation on the plane. In particular, there are six symmetry axes in the plane. Three of them correspond to the reflection ν in D_3 and its conjugates by rotations of angles $2\pi/3$ and $4\pi/3$, and three others correspond to the reflection $S\nu$ and its conjugates by $2\pi/3$ and $4\pi/3$. Finally, we have shown that this bifurcation problem is essentially equivalent to the steady-state bifurcation with D_6 symmetry in the plane. It is well-known that it has two different types of solutions, which correspond to the two different types of symmetry axes for the hexagon, and which have opposite stability.

In order to exhibit the pattern of the bifurcated magnetic fields, we need to determine, from the knowledge of $\mathbf{b}_c$, the eigenvectors that are fixed under reflections of the two types in O^-/D_2. The reflections can be chosen as follows (as elements of $O(3) \times \{S, I\}$): (i) the reflection ν corresponds to reflection through the plane $x = y$ in $\mathbf{R}^3$; (ii) the transformation $S\nu$. Note that $\mathbf{b}_c^+ = \mathbf{b}_c + \nu\mathbf{b}_c$ is ν invariant, while $\mathbf{b}_c^- = \mathbf{b}_c - \nu\mathbf{b}_c$ is $S\nu$ invariant. By forming these two vectors, we therefore determine the pattern of the two types of bifurcated solutions. These are shown in Figure 11. Inspection of these pictures shows that there is a subtle but important difference between the two types of solutions.

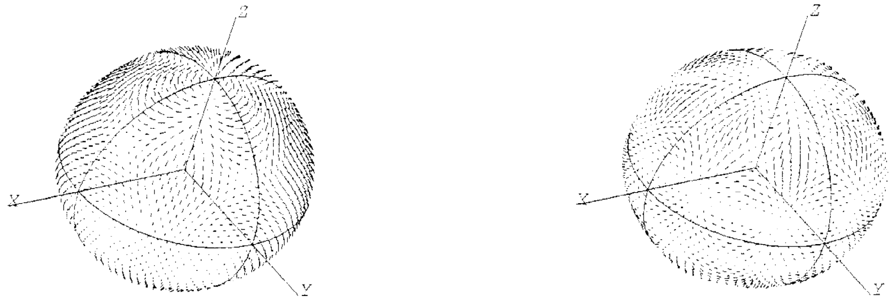


Fig. 11. $\eta = 0.4$, tetrahedral basic flow. Bifurcated magnetic fields on the sphere $r \sim 0.6$. (a) $\mathbf{b}_c^+$ solution with reflectional symmetry ν; (b) $\mathbf{b}_c^-$ solution with symmetry $S\nu$. Both solutions keep the D_2 symmetry of the basic flow (i.e., rotation by π around each coordinate axis).

As in the previous cases, neither one nor the other type of solutions does allow a drift, because in each case the group Σ combines rotations and reflections, and therefore $N(\Sigma)$ is a finite group.

4. Discussion and Concluding Remarks

We set up a versatile numerical code for the linear stability analysis of the dynamo problem, allowing us (i) to input any kind of velocity **u**, expressed in a series expansion of generalised spherical functions; (ii) to take full account of the symmetries that come to play in the problem. Although the code solves only the linear problem, it can provide the main qualitative properties of the symmetry-breaking dynamo bifurcation. Indeed, it determines the form of the spherical components of the critical eigenvectors in terms of generalised spherical functions; hence, it allows us to determine, in a relatively easy way, the action of the symmetry group (i.e., isotropy subgroup) of the basic velocity field on the critical eigenspace.

Once this action is known, it is straightforward to apply the formalism of symmetry-breaking bifurcation theory in order to extract the most significant qualitative information about the bifurcated magnetic fields. On the examples we have studied, we have found that such information is by no means intuitive. For example, in both cases $\eta = 0.1$ and 0.3 with an axisymmetric input flow, the bifurcation breaks the rotational symmetry in the same way. This is in accordance with [12], where it was shown that for an axisymmetric vector field, the magnetic field behaves like $e^{i\varphi}$. In the second case, the equatorial plane reflection of the magnetic field is conserved; hence, **b** is a quadrupole (see [13]). On the contrary, in the first case ($\ell_o = 1$), the equatorial symmetry is destroyed. We have also been able to predict whether the new solutions do typically drift, like rotating waves along their group orbit. In all the examples we have studied, there is no drift.

To complete the bifurcation diagram, one should compute the leading part of the branching solutions. However, this is a very difficult task, because it requires the computation of the bifurcation equations, i.e., the resolution of the full, nonlinear dynamo equations (1) to (5). This is difficult not only because the equations are complicated, with a large number of unknowns, but also because the critical eigenvectors are known only up to a relatively crude numerical approximation. By concentrating on the more qualitative aspects, we have been able to predict many features of the bifurcated solutions. It might happen for example that the bifurcating branch starts off in the "wrong direction" (subcritically), in which case this dynamo solution would be nonobservable (unstable) near onset.

Another aspect of our analysis is that it has enlightened us as to the role of the symmetries in this problem, especially of S: $\mathbf{b} \mapsto -\mathbf{b}$. This role has also been noticed in a recent paper by Knobloch [15]. We have shown for example that its presence excludes drifting of the solutions in the case $\ell_0 = 1$. One can interpret this fact as follows. A binary symmetry κ (rotation by π or reflection through a plane) acts on the bifurcated solution either trivially or as $-Id$. In the latter case, κ is of course not a symmetry of that solution; however, $S\kappa$ is. This makes the isotropy group of the solution large enough so that its normalizer is finite-dimensional (thanks to Lemma 1). If this symmetry could be slightly broken, while keeping the other symmetries in the model, then, as a consequence of this discussion, one should expect a slow drift to occur after bifurcation. Another instance in which a drift can occur is when the spherical symmetry is not perfect. In particular, if a small uniform rotation of the shell is now allowed, then the dynamo bifurcation always leads to rotating waves, because the bifurcated magnetic field is never axisymmetric (and reflectional symmetries are broken by the rotation). Then we know that a drift along the $SO(2)$ orbit should typically result. Note that, on the contrary, axisymmetric and therefore stationary purely convective flows still do exist in this case [4], [6]. It must be noted here that the symmetry S was not explicitly taken into account in a previous paper about a similar approach to the dynamo problem, but in the case of planar convection [25]. This paper had several shortcomings that we have overcome in the present work (in particular by considering a spherical domain).

There is a priori no restriction for the velocity field that we input in our model, except of course that it should be stationary (for a different approach allowing for time-dependent velocities, see [21]). We could, for example, choose initial flows assuming the form of a rotating wave in a rotating spherical shell. Indeed, such flows are stationary in a suitable rotating frame, and the eigenvalue problem for the magnetic disturbance is still well-defined. We can claim with a high degree of confidence that the axisymmetric velocities when $\eta = 0.1$ and 0.3 lead to a dynamo effect, which however breaks the axisymmetry. This is coherent with the "antidynamo" theorem stated by Cowling in 1934, saying that an axisymmetric velocity cannot give rise to an axisymmetric magnetic field with the same axis of symmetry (see [19]). In the case $\eta = 0.3$, we also found that the critical parameter value δ_c is larger for an axisymmetric initial flow than for a nonaxisymmetric one. This is consistent with the idea that a dynamo is more likely generated when the flow is more complicated.

In all the examples we have considered here, the bifurcation is a steady-state bifurcation. However, we expect Hopf bifurcation in the case when a rotating spherical shell is considered, as happens in planar convection [25].

Appendix: Dynamo Equation in Terms of Generalised Spherical Functions

The dynamo equation has the form

$$\Delta\, b \;+\; \delta\, \nabla \times\; (v \;\wedge\; b) \;=\; \lambda\, b,$$
$$\nabla . b \;=\; 0.$$

In the spherical basis (e_r, e_θ, e_z), the expression for the Laplacian reads

$$(\Delta b)^o \;=\; \Delta b^o \;-\; \frac{2}{r^2} b^o - \;\frac{i\,\sqrt{2}}{r^2}\left(\frac{\partial b^+}{\partial \theta} \;+\; \frac{i}{\sin\theta}\; b^+ \;+\; \mathrm{cotg}\theta\; b^+ \;+\; \frac{\partial b^-}{\partial \theta} \;-\; \frac{i}{\sin\theta}\; b^- \;+\; \mathrm{cotg}\theta\; b^-\right),$$

$$(\Delta b)^+ \;=\; \Delta b^+ \;-\; \frac{2\,i}{r^2}\frac{\cos\theta}{\sin^2\theta}\;\frac{\partial b^+}{\partial \varphi} \;-\; \frac{1}{r^2\;\sin^2\theta}\; b^+ \;-\; \frac{i\,\sqrt{2}}{r^2}\left(\frac{\partial b^o}{\partial \theta} \;-\; \frac{i}{\sin\theta}\;\frac{\partial b^o}{\partial \varphi}\right),$$

$$(\Delta b)^- \;=\; \Delta b^- \;+\; \frac{2\,i}{r^2}\frac{\cos\theta}{\sin^2\theta}\;\frac{\partial b^-}{\partial \varphi} \;-\; \frac{1}{r^2\;\sin^2\theta}\; b^- \;-\; \frac{i\,\sqrt{2}}{r^2}\left(\frac{\partial b^o}{\partial \theta} \;+\; \frac{i}{\sin\theta}\;\frac{\partial b^o}{\partial \varphi}\right),$$

with $\Delta f \;=\; \frac{\partial^2 f}{\partial r^2} \;+\; \frac{2}{r}\,\frac{\partial f}{\partial r} \;+\; \frac{1}{r^2}\,\frac{\partial^2 f}{\partial \theta^2} \;+\; \frac{1}{r^2}\,\mathrm{cotg}\theta\;\frac{\partial f}{\partial \theta} \;+\; \frac{1}{r^2\;\sin^2\theta}\,\frac{\partial^2 f}{\partial \varphi^2}$. This Laplacian expressed in terms of generalised spherical functions reads

$$(\Delta b)^o \;=\; \sum_1 T^l_{o,m}\left(D_l b^l_{o,m} - \frac{2}{r^2}\; b^l_{o,m} - \frac{\sqrt{2}}{r^2}\alpha_l \left(b^l_{+,m} + b^l_{-,m}\right)\right),$$

$$(\Delta b)^+ \;=\; \sum_1 T^l_{+,m}\left(D_l b^l_{1,m} - \frac{\sqrt{2}}{r^2}\;\alpha_l\; b^l_{o,m}\right),$$

$$(\Delta b)^+ \;=\; \sum_1 T^l_{-,m}\left(D_l b^l_{-1,m} - \frac{\sqrt{2}}{r^2}\;\alpha_l\; b^l_{o,m}\right).$$

The complete eigenvalue problem which includes the curl part takes the form

$$\lambda\; b^l_{o,m} \;\;=\; D_l\; b^l_{o,m} \;+\; \frac{2}{r}\; D b^l_{o,m} \;+\; \frac{2}{r^2}\; b^l_{o,m} \;+\; \delta\; \sum_4\; z_{m_2} \left(\frac{\sqrt{2}}{2}\;\alpha_l\; C^l_2\;\frac{u^{l_2}_o}{r}\;\left[b^{l_1}_{+,m_1} \;+\; (-1)^{l_1+l_2+l}\; b^{l_1}_{-,m_1}\right] -\frac{\sqrt{2}}{2}\;\alpha_l\; C^l_3\;\frac{u^{l_2}_+ \;+\; (-1)^{l_2+l_1+l}\; u^{l_2}_-}{r}\; b^{l_1}_{o,m_1}\right),$$

$$\lambda\, b^l_{+,m} \;=\; D_l\, b^l_{+,m} \;-\; \frac{\sqrt{2}}{r^2}\,\alpha_l b^l_{o,m} \;+\; \delta \sum_4 z_{m_2}$$
$$\left(\frac{\sqrt{2}}{2}\,\alpha_l\, C^l_1 \left[\frac{u^{l_2}_+}{r}\, b^{l_1}_{-,m_1} \;-\; (-1)^{l_1+l_2+l}\; \frac{u^{l_2}_-}{r} b^{l_1}_{+,m_1} \right] \right.$$
$$\left. -\, C^l_3 \frac{1}{r} \frac{\partial \left(r\, u^{l_2}_o\, b^{l_1}_{+,m_1} \right)}{\partial r} \;+\; C^l_2 \frac{1}{r} \frac{\partial \left(r\, u^{l_2}_+\, b^{l_1}_{o,m_1} \right)}{\partial r} \right),$$

$$\lambda\, b^l_{-,m} \;=\; D_l\, b^l_{-,m} \;-\; \frac{\sqrt{2}}{r^2}\,\alpha_l b^l_{o,m} \;+\; \delta \sum_4 z_{m_2}$$
$$\left(-\,\frac{\sqrt{2}}{2}\,\alpha_l\, C^l_1 \left[\frac{u^{l_2}_+}{r}\, b^{l_1}_{-,m_1} \;-\; (-1)^{l_1+l_2+l}\; \frac{u^{l_2}_-}{r}\, b^{l_1}_{+,m_1} \right] \right.$$
$$\left. +\, (-1)^{l_1+l_2+l} \left\{ -\; C^l_3 \frac{1}{r} \frac{\partial \left(r\, u^{l_2}_o\, b^{l_1}_{-,m_1} \right)}{\partial r} \;+\; C^l_2\, \frac{1}{r} \frac{\partial \left(r\, u^{l_2}_-\, b^{l_1}_{o,m_1} \right)}{\partial r} \right\} \right),$$

together with the divergence-free condition

$$0 = \frac{\sqrt{2}}{2}\,\alpha_l\, \frac{1}{r}\, \left(b^l_{+,m} \;+\; b^l_{-,m} \right) + D b^l_{o,m} \;+\; \frac{2}{r} b^l_{o,m},$$

where $\sum_4$ stands for $\sup(m, |l_1 - l_2|) \;=\; l$ and $m_1 + m_2 \;=\; m$ for $-l_1 \;\le\; m_1 \;\le\; l_1$, $-l_2 \le m_2 \le l_2$, and the unknowns $b^\ell_{0,m}, b^\ell_{\pm,m}$ statify the the boundary conditions

$$b^l_{o,m} \;=\; \frac{(r\; b_{\pm,m})}{\partial r} \;=\; 0, \qquad \text{for } r \;=\; \eta \text{ and } 1.$$

Notice that the equation satisfied by the unknown $(b_{+,m} \;+\; b_{-,m})$ can be deduced from the these of $b_{o,m}$. Consequently, we solve the generalised eigenvalue problem for the two unknowns $b_{o,m}$ and $(b_{+,m} \;+\; b_{-,m})$.

$$\lambda\, b^l_{o,m} \;\;=\; D_l\, b^l_{o,m} \;+\; \frac{2}{r}\, D b^l_{o,m} \;+\; \frac{2}{r^2}\, b^l_{o,m} \;+\; \delta \sum_4 z_{m_2}$$
$$\left(\frac{\sqrt{2}}{2}\,\alpha_l\, C^l_2\, \frac{u^{l_2}_o}{r}\, \left[b^{l_1}_{+,m_1} + (-1)^{l_1+l_2+l}\, b^{l_1}_{-,m_1} \right] \right.$$
$$\left. -\frac{\sqrt{2}}{2}\,\alpha_l\, C^l_3 \frac{u^{l_2}_+ \;+\; (-1)^{l_2+l_1+l}\, u^{l_2}_-}{r}\; b^{l_1}_{o,m_1} \right),$$

$$\lambda\,(b^l_{+,m} - b^l_{-,m}) = D_l\,(b^l_{+,m} - b^l_{-,m}) + \delta \sum_4 z_{m_2}$$

$$\left(\frac{\sqrt{2}}{2}\,\alpha_l\, C^l_1 \left[\frac{u^{l_2}_+ - (-1)^{l_1+l_2+l}\, u^{l_2}_-}{r}\,(b^{l_1}_{+,m_1} + b^{l_1}_{-,m_1}) \right.\right.$$

$$\left. - \frac{u^{l_2}_+ + (-1)^{l_1+l_2+l}\, u^{l_2}_-}{r}\,(b^{l_1}_{+,m_1} - b^{l_1}_{-,m_1}) \right]$$

$$-C^l_3 \frac{1}{r} \frac{\partial \left(r\, u^{l_2}_o \left[b^{l_1}_{+,m_1} - (-1)^{l_1+l_2+l} b^{l_1}_{-,m_1} \right] \right)}{\partial r}$$

$$\left. + C^l_2\, \frac{1}{r} \frac{\partial \left(r \left[u^{l_2}_+ - (-1)^{l_1+l_2+l} u^{l_2}_- \right]\, b^{l_1}_{o,m_1} \right)}{\partial r} \right),$$

which allows us to eliminate the $b^l_{+,m} + b^l_{-,m}$ term thanks to the divergence-free condition.

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