STUDY OF THE CONTINUATION METHODS FOR PERIODIC ORBITS OF THE FROESCHLÉ MAP

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Abstract

In this paper we compare different continuation methods for the searching of periodic orbits of the Froeschlé map, which is a four dimensional symplectic map. We find relations between the stability of the periodic orbits and the problem to solve the system given in the continuation method.

1. INTRODUCTION

For many mechanical problems, of a few degrees of fredom, the question of stability is related to the problem of finding periodic orbits. Actually for a Hamiltonian system of two degrees of freedom this relation is well known. However, for higher numbers of degrees of freedom there are only few results in this direction.

A common procedure to study periodic orbits is to reduce the flow in the phase space to a discrete system using the Poincaré section. In this work, we are interested in a Hamiltonian flow in a six dimensional space which will be reduced to a symplectic map in a four dimensional space.

Most of three degrees of freedom Hamiltonian systems have some symmetries, such that there are periodic orbits which inherited some of the symmetries of the original system. These orbits are known as symmetric periodic orbits.

A symplectic map has the same number of symmetries than the original Hamiltonian flow. The symmetric periodic orbits have the property that some images of the orbit belong to some two dimensional planes, these planes are invariant under the symmetries of the system. This fact simplifies the task of finding symmetric periodic orbits, because the searching of these orbits is restricted to a two dimensional plane. This procedure is equivalent to find the roots of a function in a two dimensional domain. When the system also depends upon parameters the roots of the function mentioned above will also depend on these parameters.

We shall study a particular but important symplectic map which is known as Froeschlé map, which is defined in $(\mathsf{S}^1 \times \mathsf{IR}) \times (\mathsf{S}^1 \times \mathsf{IR})$ by:

$$f(\mathbf{x}) = \begin{cases} \overline{y}_i = y_i - \frac{K_i}{2\pi}\sin(2\pi x_i) - \frac{\lambda}{2\pi}\sin(2\pi(x_1+x_2)) \ , \\ \\ \overline{x}_i = x_i + \overline{y}_i \pmod 1 \ , \end{cases} \qquad i = 1,2 \tag{1}$$

where $\mathbf{x} = (x_1, y_1, x_2, y_2)$,$x_i \in \mathsf{S}^1$ and $y_i \in \mathsf{IR}$. This map depends on three parameters K_1, K_2, λ and has eight symmetry planes, these two dimensional planes are defined by:

$$\mathrm{SR}_{(m_1,m_2)} = \left\{(x_1, y_1, x_2, y_2) : x_1 = \frac{m_1}{2}, x_2 = \frac{m_2}{2}\right\} \quad , \tag{2}$$

$$\mathrm{TSR}_{(m_1,m_2)} = \left\{(x_1, y_1, x_2, y_2) : y_1 = 2(x_1 + \frac{m_1}{2}), y_2 = 2(x_2 + \frac{m_2}{2})\right\} \quad ,$$

with $m_1, m_2 = 0, 1$

The numerical procedure to find symmetric periodic orbits of period n, is the following:

1. We take a point in one of the symmetry planes.

2. This point is iterated $n/2$ times (or $(n+1)/2$ if n is odd) and we record the distance of the image of this point to the other symmetry plane. If the distance is zero, the point belongs to a symmetric periodic orbit.

3. If the distance is not zero, we must find another point which diminish this value of the distance. This procedure, is equivalent to find a root of a function, therefore it is possible to use a root finder, for example a Newton–like method.

The existence of this kind of orbits for any period and any value of the parameters, is discussed in (Kook & Meiss, 1989), where they used a variational method.

2. TURNING POINTS

We are interested in following symmetric periodic orbits for a range of values of the parameters, and a continuation method is suitable to do this. The procedure to calculate the solution branches is fully explained in (Olvera & Vargas, 1994). Taking λ as our continuation parameter (such that K_1 and K_2 are function of λ), we can find a solution branch in the coordinate–parameter space (see figure 1). It is posible that one of the branches has a turning point and returns to the initial value of the parameter.

The existence of multiple solution branches gives the posibility that the continuation method may find numerical difficults and then jump from one branch to another one. There are many reasons for this behaviour, for example, the reduction of the convergence basin of the Newton–like method or the atempt to obtain a periodic orbit beyond the turning point, among others.

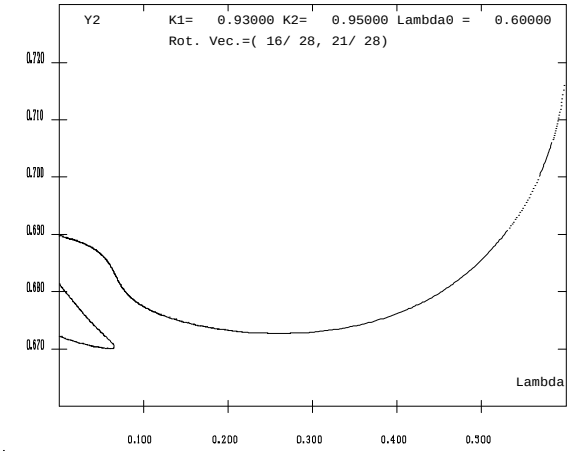

Figure 1: Solution branches of the periodic orbit with rotation vector $(\frac{16}{28}, \frac{21}{28})$.

Therefore it is necesary to use a robust continuation method in order to avoid the mentioned problems. Any continuation method is made out of a predictor and a corrector part. Let x_0 be a root of a function $F(x, \lambda)$ when $\lambda = \lambda_0$, the task of an Euler–predictor is to estimate the value of x_1 root of $F(x, \lambda)$, when $\lambda = \lambda_1$, by the equation:

$$\hat{x}_1 = x_0 - \left(\frac{\partial F(x,\lambda)}{\partial x}\right)^{-1} \left(\frac{\partial F(x,\lambda)}{\partial \lambda}\right) \Delta\lambda \quad , \tag{3}$$

where $\Delta\lambda = \lambda_1 - \lambda_0$. The corrector part, a Newton–like method, uses the estimated value $\hat{x}_1$ to approach the actual root x_1 of $F(x, \lambda_1)$. The step size $\Delta\lambda$ should be suficiently small in order to ensure that the estimated value $\hat{x}_1$ belongs to the attracting basin of the root x_1, otherwise the corrector procedure could diverge.

The existence of turning points in the solution branches is a problem for the predictor method, because the jacobian $(\partial F/\partial x)$, in equation (3), is singular at this point. In order to avoid the singular values of the jacobian, it is necesary to add a new variable and a new equation. It is very convenient to use, as a new variable, the arclenght σ of the branch. In this way, we form a new system of equations, $G(x, \lambda, \sigma) = 0$, and we can estimate the value of x and λ for any value of σ. The dimension of the domain of G is $n + 1$, where n is the dimension of the domain of F.

There are many variants to define the equation $G(x, \lambda, \sigma)$ and the variable σ. Keller (Keller, 1987) proposed the following system:

$$G(x, \lambda, \sigma) = \begin{cases} F(x, \lambda) = 0 \\ ||x - x_0||^2 + |\lambda - \lambda_0|^2 - |\sigma - \sigma_0|^2 = 0 \end{cases} \tag{4}$$

In this case, the variable σ is known as the semi–arclenght of the curve,and x_0, λ_0 and σ_0 are constants. The jacobian of $G(x, \lambda, \sigma)$ is now non–singular in the turning points. Nevertheless this fact does not mean that there are not singular values of the jacobian of G; there may exist singular values of this jacobian, and since they depend on the constants

x_0, λ_0 and σ_0, we can remove a singular point by changing the values of this constants. We call this a reescaling procedure.

The predictor method can be improved using a better interpolation method. In our case we included an Adams–Bashforth method of order four, this means that we use the last four points in order to predict the next point. This is particularly usefull when the curvature of the path, changes as in the presence of turning points. A similar procedure is discussed in (Lundberg & Poore, 1991).

For the corrector procedure there are also several methods that can be used. Some of them are less expensive than others; e.g. those which do not compute the inverse of the jacobian matrix at each iteration, in this class we have Broyden's method, which decreases the computation time, since in this case the inverse of the jacobian matrix is replaced by a new matrix that is easier to compute.

3. DIFFICULTIES IN THE NUMERICAL METHOD

The continuation methods which increase the dimension of the problem, like the semi–arclenght method, are not always a better choice over the simpler continuation method. We have found that the calculation of the root of $F(x,\lambda)$ far from the turning points and secondary bifurcation points, implies a larger computation time of the higher dimensional continuation methods over the simpler continuation method.

An additional problem is the increase of the numerical error obtained in the computation, for example the computation of the inverse of the jacobian matrix can give a biger error in the higher dimensional continuation method. A quantitative way to evaluate this phenomena is by use of the condition number of a matrix, which is given by $\mathcal{K}(A) = ||A||||A^{-1}||$, this number gives an idea of the amout that the error is magnify when the inverse of a matrix is calculated. This number can grow with the dimension, and the tipical example is given by the so called Hilbert matrix (Atkinson, 1989), which is defined by $H_n(i,j) = \left(\frac{1}{i+j-1}\right)$ and the condition number for some values of n are:

n	$\mathcal{K}(H_n)$	n	$\mathcal{K}(H_n)$
3	5.24×10^2	7	4.75×10^8
4	1.55×10^4	8	1.53×10^{10}
5	4.77×10^5	9	4.93×10^{11}
6	1.50×10^7	10	1.60×10^{13}

This condition number tell us that we can expect a loss of two digits in the calculation of the inverse when we increase the dimension of the matrix by one.

In general we can expect more dificulties if we increase the dimension of the problem, therefore we would like no to use the semi–arclenght method unless it is strictly necessary.

A consecuence of increasing the dimension, is the possible reduction of the accuracy of the predictor procedure, which makes it necessary to reduce the step size of the semi–arclenght σ; otherwise the corrector procedure would not converge, making the continuation

of the branch a very slow procedure.

Another problem, when the dimension is increased, is that the size of the convergence basing of the Newton–like method is reduced, making the procedure more expensive. A simple example of the reduction of the convergence basin with respect the dimension can be ilustrated by the function $f(\mathbf{x}) = \mathbf{x}e^{r^2}$ where $\mathbf{x} \in \mathbb{R}^n$ and $r = ||\mathbf{x}||$. In this case, the radius of the convergence region is proportional to $1/n^2$.

Therefore, it is clear the convenience of using the more complex method (semi–arclenght method) only in the neighbourhood of turning points or of secondary bifurcation points. The main problem is to predict the location, in the branch solution, of those problematic points. For this fact we recall that our system is Hamiltonian, then we can use some properties of these systems in order to obtain more information about the location of the those points.

4. PROPERTIES OF A SYMPLECTIC MAP

The first property to be mentioned is that any periodic orbit of our symplectic map is created by a bifurcation of the saddle–node type, therefore two periodic orbits are created at each time. This means that secondary bifurcation points do not occur. Thus the only problematic points will be turning points.

Some more information is obtained from the eigenvalues of the jacobian matrix of the periodic orbit, since in a bifurcation point at least two of the eigenvalues are one. Therefore to avoid the turning points we have to be sure that the eigenvalues are far from 1 in the complex plane; we can assure this, if one of the following condition obeys:

- C1. $|\lambda_1| > 1$ and $|\lambda_2| > 1$.
- C2. $|\lambda_1| > 1$ and $|\mathrm{Imag}(\lambda_2)| > 0$.
- C3. $|\mathrm{Imag}(\lambda_1)| > 0$ and $|\mathrm{Imag}(\lambda_2)| > 0$.

And this is true, because a sympletic map has the property that if λ is an eigenvalue then λ^{-1}, $\bar{\lambda}$ and $\bar{\lambda}^{-1}$ are also eigenvalues.

Due to the continuity of the eigenvalues upon parameters, we can expect that if they are not close to 1, then in a neighbourhood of the values of the parameters the same behaviour will hold, and therefore we will know that there are not bifurcation points. And in this neighbourhood we can use a simpler continuation method without being afraid of finding bifurcation points.

This idea is not very expensive since the eigenvalues are rather easy to find, since the characteristic polynomial of the jacobian matrix is a palindrome, therefore the four eigenvalues are obtained by solving only two second order algebraic equations (Howard & MacKay, 1986).

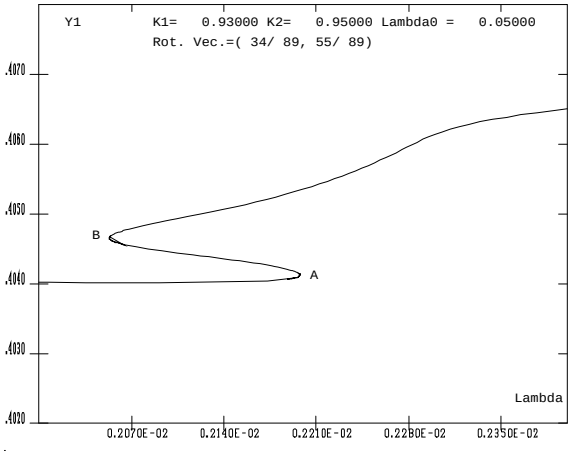

Figure 2: Solution branch of the periodic orbit with period 89. The branch is in the plane (λ, y_1).

5. EXAMPLES

Now, we are going to examine a set of particular periodic orbits of the Froeschlé map, in order to verify the differences of the two continuation methods. We select a set of symmetric periodic orbits, whose periods are the Fibonacci numbers (5, 8, 13...). The continuation method is started from $\lambda = 0$ because map (1) can be written as a pair of two dimensional maps. For these maps, it is easy to find periodic orbits, therefore we use these points as initial points in the continuation method (Olvera & Vargas, 1994). The parameters K_1 and K_2 are taken as functions of λ, defined in the following form:

$$K_i(\lambda) = K_i^0 \left(\frac{\lambda_0 - \lambda}{\lambda_0} \right)^{1/2} \qquad i = 1, 2$$

where $K_1^0 = 0.93$, $K_2^0 = 0.95$ and λ_0 is chosen acording to the period of the orbit. The range of λ is $[0, \lambda_0]$, and we compute the solution branches in this range of λ.

As an example we use an orbit of period 89, in this case we take $\lambda_0 = 0.05$. The solution branch has two turning points and in this case the branch is defined for all the range of λ. Figure 2 shows a small part of the branch when $\lambda \in [0.002, 0.0025]$, where we can observe two turning points, A and B in the graph. We compute the condition number of the matrix $(\partial F/\partial x)$ of equation (3) and $(\partial G/\partial x)$ of equation (4). The first case corresponds to the simpler continuation method and the second one to the semi–arclenght method. Figure 3 shows the graphs of $\mathcal{K}(\partial G/\partial x)$ and $\mathcal{K}(\partial F/\partial x)$. It is clear that the simpler method is more efficient than the semi-arclenght method for almost all values of λ. The differences of the values of the condition numbers is of three orders of magnitude, this means that the predictor method of the simpler case is more robust, because we can compute numerically the inverse matrix of $(\partial F/\partial x)$ in a more exact way. We can use larger step sizes of λ in this simpler method because the predictor procedure gives very accurate values for the next value of x. In this way we can justify the use of an Adams–Bashforth procedure in order

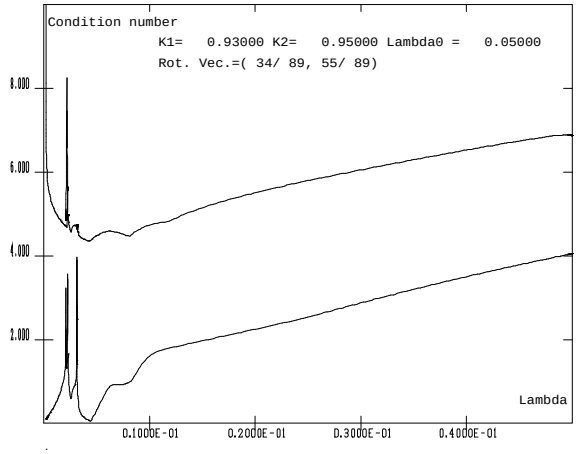

Figure 3: The upper graph represents the logarithm of $\mathcal{K}(\partial G/\partial x)$ vs. λ, the lower graph shows $\mathcal{K}(\partial F/\partial x)$ vs. λ.

to increase the accuracy of the predictor. This procedure is not recomendable to be used in the semi–arclenght method since the accuracy that we gain using the Adams–Bashforth procedure can be lost by the overall error when we compute the inverse of the matrix $(\partial G/\partial x)$.

There is a range of λ where the simpler method has a bad behaviour in the predictor procedure, due to the existence of singular values of the jacobian. Figure 4 shows two values of λ where the condition number of $(\partial F/\partial x)$ grows extremely fast (see point A and B), these points correspond to two turning points of the solution branch (points A and B in figure 3). Nevertheless, the condition number of $(\partial G/\partial x)$ does not increase its value in this two points. Therefore the semi–arclenght method is the only possibility to continue the branch across these two points.

Examining figure 4, we also find two points where the condition number of $(\partial G/\partial x)$ has an irregular behaviour (see points C and D). These singular values do not correspond to any turning point of the solution branch. These points can be removed if we redifine the values of the constants x_0, λ_0 and σ_0 in equation (4), as we mention before.

The main problem with these singular values C and D, is that they are normally located very close to the turning points. Then the rescaling procedure must be done with some care, in order to avoid the relocation of the singular points ahead on the solution branch, rather the idea is to send them back, in such a way that we can not find them again when we are moving forward on the solution branch.

We find another singular value of $\mathcal{K}(\partial G/\partial x)$ when $\lambda \to \lambda_0 = 0$ (see figure 3). In this case we can anticipate this behaviour, because the last row of $(\partial G/\partial x)$ is given by $(x - x_0, y - y_0, \lambda - \lambda_0)$, therefore when $\lambda \to \lambda_0$ the last row tends to $(0, 0, 0)$ and in the inverse matrix we have elements of the form $1/(y - y_0)$.

The behaviour of the eigenvalues is shown in figures 5 and 6. The logarithm of the real

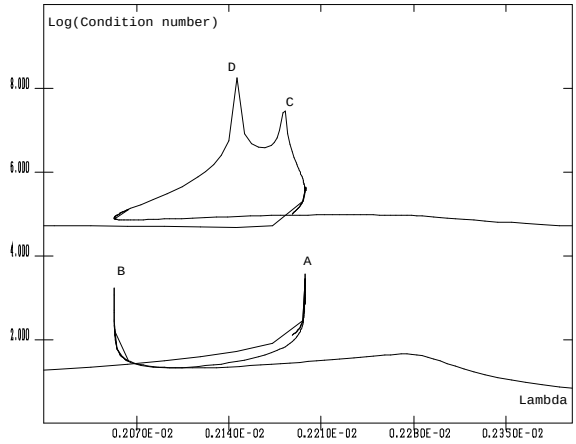

Figure 4: Detail of figure 3.

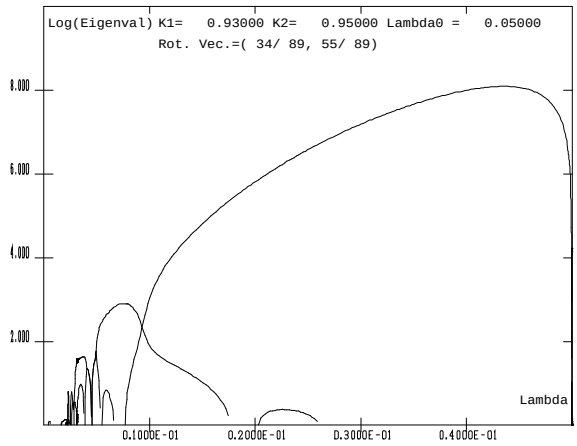

Figure 5: Logarithm of the real part of the eigenvalues of the periodic orbit.

part of the eigenvalues is given in figure 5, where we can see that one of the eigenvalues grows rapidly when $\lambda > 0.01$. Also the magnitude of the imaginary part of two of the eigenvalues is very close to 1 in the range $\lambda \in [0.03, 0.045]$.

Thus we can say that the simpler method is more efficient when the eigenvalues obey the conditions C1, C2 or C3. Therefore the eigenvalues can be used as an indicator, to switch from the simpler to the more complex method, and viceversa.

6. CONCLUSIONS

We show in our previous discusion that the more complex continuation method is not necesarilly the best way to compute the solutions. This kind of methods are necessary only in the neigbourhood of the turning points, and this neighbourhood can be predicted by the behaviour of the eigenvalues using conditions C1, C2 or C3.

We can also say that the Adams–Bashforth method can be used in an optimal form

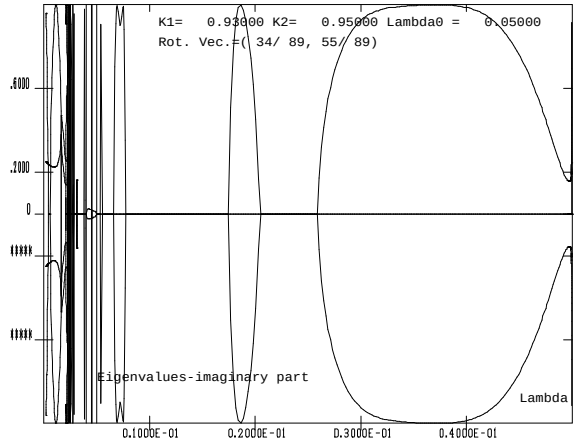

Figure 6: Imaginary part of the eigenvalues of the periodic orbit.

when we can be sure that in the differential equation to be solved:

$$\dot{x} = \left(\frac{\partial G}{\partial x}\right)^{-1} \left(\frac{\partial G}{\partial \lambda}\right)$$

the inverse matrices can be calculated with enough accuracy, otherwise a cheaper method, as Euler, given by equation (3) will be more adecuate.

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