Some self-consistent methods in the mechanics of composite materials.

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Composite materials that consist of two or more constituents perfectly bounded together are finding an increasing role in modern technology, including the manufacture and oil industries. This explains the interest on the investigation of a variety of macroscopic (or overall) mechanical and other properties of such materials and their response to external actions. Among the many kinds of composite materials, there is an important class, the so-called particulate composites, that consist of a homogeneous matrix and embedded inclusions which may be occupied by another material or may be empty (includes the cases of gas or fluid filled cracks). The main difficulty in the theoretical description of the overall properties of such materials lies in the necessity of taking into account the interaction of many randomly positioned inclusions. There is group of methods in theoretical physics, named self-consistent methods that allow the approximate solution of this problem.

The aim of these lectures is to describe some of these methods in detail and to present the main results that can be obtained with the help of them in the prediction of the overall elastic (including static and dynamic) properties of matrix composite materials, including the case of cracks or fractures of interest in rock mechanics.

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Preface

The study and the understanding of phenomena, whether physical or biological, in which inhomogeneous media are involved is very important from various points of view. Mathematically, it presents interesting questions. Technologically, it has an enormous range of applications in, for instance, the manufacture and oil industries. The lecture notes presented here provide an introduction to a group of mathematical techniques which deal with problems of wide interest. These lectures may be useful to applied mathematicians, physicists, material scientists, engineers, and mathematical biologists.

One of the aims of the Proyecto Universitario de Fenómenos Nolineales (FENOMEC) at Universidad Nacional Autónoma de México (UNAM) is to provide a forum were specialists, graduate students and active researchers, from different disciplines share and interact in a common activity. It is for this reason that this is one of the many activities in the area of composite materials that FENOMEC is encouraging. Interest from many quarters has been shown, such as Institutos de Física, Ingeniería, IMAS, de Investigaciones en Materiales, Centro de Instrumentos from UNAM; Facultad de Ingeniería, Universidad Autónoma del Estado de México and Instituto Mexicano del Petróleo.

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I N D I C E

1 Introduction

1.1 The Aim of Micromechanics

Continuum mechanics deals with ideal homogeneous materials. Its aim is to describe their response to external loadings using appropriate constitutive relations. These relations generally are determined by means of macroscopic experiments without any micro-structural consideration. It is well known that all real materials are microscopically inhomogeneous, even if they appear homogeneous at some natural scale of observation. Hence, a description of any material in terms of continuum mechanics is an approximation, and any experimental determination of constitutive behavior yields, in fact, a relationship between the "overall" properties measured in the experiment. This observation leads us to a fundamental problem of science and technology, concerning "micro-macro" interconnection, i.e. a proper determination of the macroscopic (or "large" scale) behavior of a medium, which exhibits microscopic (or "small" scale) heterogeneity, on the base of the appropriate and available micro-structural information. The oldest problem of such a type, fundamental in statistical physics, is the description of matter in terms of its molecular constituents. Here "small" corresponds obviously to molecular dimensions.

Micro-mechanics, in general, deals with heterogeneous media for which "small" has certain intermediate dimensions, but is small in macro-scale. The length l is connected with the characteristic size of heterogeneities in the medium, say, with the mean radius of inclusions, voids, fibers, the size of a crystallite in polycrystalline aggregates etc. The aim of micro-mechanics is just to relate the microscopic structure of heterogeneous media, characterized with the above mentioned scale l, to the macroscopic behavior. The basic idea is that of homogenization, which consists in the replacement of a sample of a micro-inhomogeneous solid by a homogeneous one, which from a macroscopic point of view "behaves" in the same manner as this sample.

The important class of heterogeneous media, extensively treated by micro-mechanics, are the composites – man-made mixtures of two or more constituents, firmly bound together. Much interest in the investigation and modeling of physical and mechanical properties of composite materials is connected with the constantly increasing area of their application in modern technology. In many respects composite materials proved to be superior to the common homogeneous materials: first of all they have superior physical and mechanical properties, secondly, it is possible to design the composite structure and create materials with properties prescribed in advance, optimal for the operation conditions of the whole structure. Comprehensive investigations of the influence of micro-structure on the whole specter of physical and mechanical properties of composite materials are necessary for the solution of the latter problem.

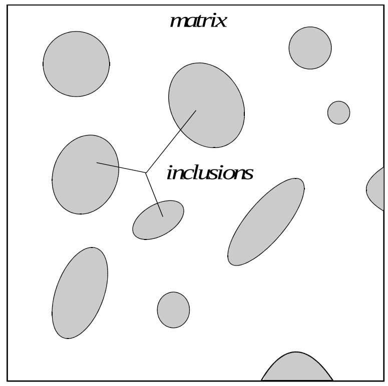

Fig. 1

Between the various kinds of composite materials there is an important class of so-called matrix composites. The micro-structure of such composites usually has the following form: it consists of a homogeneous matrix and a set of filling particles (i.e. inclusions) uniformly distributed in the matrix (Fig. 1). Examples of such composites are: composites reinforced by particles of different forms and shapes, unidirectional fiber reinforced composites, laminates and hybrid composites filled with different types of particles.

As a rule the micro-structure of real composites is stochastic: random parameters characterize shapes, sizes and physical properties of inclusions. The distribution of the inclusions in the volume of the matrix is also random. As a result strain and stress in these composites are stochastic even under a deterministic external loading.

Hence, the physical and mechanical as well as geometrical characteristics of the matrix composite materials are random functions of coordinates. The problem of the strain and stress field determination in such materials is hopelessly complicated even using advanced computer techniques. But in the most practically important cases it is not necessary to know these fields in detail. Very often it is quite enough to have the information about the average response of the micro-inhomogeneous medium on the external actions. To solve this problem it is necessary to estimate the mean values (mathematical expectation) of these fields under the deterministic external loading. The solution of this problem allows us to find the effective (or overall) properties of composite materials and replace the initial inhomogeneous medium by the homogeneous one with the known deterministic constitutive law, the reaction of which to the external action will be in some sense equivalent to the response of the initial composite material. After that, the calculation of the stress and strain fields in such homogeneous material is the common classical problem of elasticity theory. The construction of the mathematical model of a homogeneous medium that is equivalent to the real composite material is the central problem of the mechanics of the media with micro-structure (so-called homogenization problem). Let us consider the main ideas of this problem in detail.

1.2 The homogenization problem

To illustrate the basic ideas of homogenization as simply as possible, consider a (linear) elastic heterogeneous medium. Let, for example, the tensile stress-strain behavior of the medium along the axis x_1 be under investigation. Imagine that a large, say, cubical specimen V with a side L is cut out from the medium; large means here that $L >> l$, where l is the above mentioned micro-scale length. Let this cube be loaded to some level and its extension measured. Then the stress component σ_{11} would be taken as load divided by the area of the cross-section L^2 and the strain component ε_{11} as the extension divided by the original length L. These two ways to find σ_{11} and ε_{11} are obvious, if the cube is homogeneous; the heterogeneity results, however, in non-homogeneous and rapidly oscillating stress and strain at the micro-scale level. The latter quantities, calculated from experiment, thus represent, in fact, averages of the actual forces and displacement in the cube. They are just the so-called volume (or spatial) averages, to be denoted by angle brackets

$$\langle \varepsilon_{11} \rangle = \frac{1}{V} \int\limits_V \varepsilon_{11}(x) dx, \quad \langle \sigma_{11} \rangle = \frac{1}{V} \int\limits_V \sigma_{11}(x) dx \tag{1.1}$$

In turn, the proportionality coefficient

$$E_{11}^V = \frac{\langle \sigma_{11} \rangle}{\langle \varepsilon_{11} \rangle} \tag{1.2}$$

defines the effective Young modulus for the specimen (along the axis x_1). This means that the cube, through the relation (1.2), is "homogenized" in the sense that

its heterogeneity is smoothed out and it is replaced by a homogeneous specimen, possessing the overall Young modulus E_{11}^V.
If we now repeat the same procedure for other cubical samples of the material, of the same size and orientation as the first, we will in general obtain slightly different values of E_{11}^V, since the interior phase geometry will not be exactly the same. This explains why the superscript "V" appeared in the notation E_{11}^V.
To define a "true" material property from such experiments, i.e. one that is connected with the heterogenous medium and its internal structure, independently of the specific choice of the sub-volumes, two natural procedures can be employed. First, perform a great number N of experiments on differently centered cubes (otherwise identical and identically oriented) and measure the appropriate values $E_{11}^{V'}, E_{11}^{V''}, ...$ for each one. Then, to suppress the specimen's dependence, it is natural to define

$$E_{11}^* = \frac{1}{N}\left(E_{11}^{V'} + E_{11}^{V''} + ...\right) \tag{1.3}$$

which is a true material property. The right-hand side of (1.3) is the simplest example of the so-called ensemble averaging. Hence, to obtain the information about the expected overall behavior of a heterogeneous medium, we should deal with the average reaction of a whole ensemble of specimens of identical shape and size, and applied identical external influence.
Let us look on the determination of the overall characteristics of the composite materials more precisely.
Let $u(x)$ be a function which describes a certain physical field in a composite. It may be the displacement vector, the temperature field, the electric potential etc. The equation describing the detailed structure of this field can be written in the form

$$(Lu)(x) = q(x) \tag{1.4}$$

where L is a linear operator depending on the properties of the matrix and inclusions, taking into account the geometric structure of the composite; $q(x)$ is the density of the field sources, x is a point in 3-D space.
The field $u(x)$ in the micro-inhomogeneous medium always can be represented as a sum of two components

$$u(x) = \langle u(x)\rangle + u'(x) \tag{1.5}$$

where $\langle u(x)\rangle$ is the slowly varying part of $u(x)$ and $u'(x)$ is the fluctuation with wavelength of about the mean distance between the inclusions and the sizes of the latter (Fig.2).

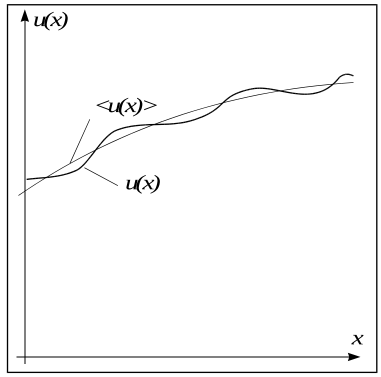

Fig. 2

The field $\langle u \rangle$ is of prime interest for applications because it characterizes the macroscopic response of the composite to the external loading. To determine the function $\langle u(x) \rangle$ let us introduce the "representative volume" of the composite V_0. This is a volume which is small enough from the macroscopic point of view (small in comparison with the wavelength of the varying macroscopic fields) and could be thus treated as a typical "point" of the heterogeneous medium under study. On the other hand, it should be large enough in the microscopical scale in order to contain a large number of inclusions and therefore to be typical "representative" for the micro-structure of the solid.
If the variation of $\langle u(x) \rangle$ inside the representative volume V_0 is negligible this function can be obtained by averaging the detailed field $u(x)$ over the volume V_0

$$\langle u(x) \rangle = \frac{1}{V_0} \int\limits_{V_0} u(x - x') dx' \tag{1.6}$$

Thus, the averaging over the volume V_0 appears here as a smoothing operation of the rapidly oscillating function.
Let us introduce the operator L^* which allows to find the function $\langle u(x) \rangle$ directly from the equation

$$(L^* \langle u \rangle)(x) = q(x) \tag{1.7}$$

without solving equation (1.4) and averaging the result over the representative volume. The operator L^* can be called the effective operator for the heterogenous medium. This operator describes the macro-process in a homogeneous medium that is equivalent to the initial micro-inhomogeneous composite material. Constructing the operator L^* is the essence of the homogenization problem.

Hence, the equivalent medium is the medium the response of which to the external action coincides in average (macroscopically) with the response of the real composite material. It follows from here, in particular, that the equivalent medium properties must not be dependent on the size of the representative volume V_0. If, as it was suggested above, the field $\langle u(x)\rangle$ is almost constant in V_0, the right-hand side of (1.5) is practically independent on the size of V_0. But in areas where $\langle u(x)\rangle$ rapidly oscillates (e.g. in the vicinity of fields concentrations or in the case of short wave propagation) the right-hand side of Eq. (1.5) will depend on the size of V_0. In this case we need to replace the average over the representative volume by the ensemble average described above.
Because of randomness of the inner structure of the composite, any physical field $u(x)$ in it can be considered as an element α of a certain probabilistic functional space. Let $\mu(\alpha)$ be a probabilistic measure in this space. Then every random field $u(x)$ is also a function of α and the mean value of $u(x)$ can be determined as

$$\langle u(x)\rangle = \int u(x,\alpha)d\mu(\alpha) \tag{1.7}$$

This definition of the field $\langle u(x)\rangle$ allows us to define uniquely both the operator L^* and the equivalent medium. If the function $\langle u(x)\rangle$ varies slowly inside the volume V_0 the two definitions of L^* (i.e. the ensemble average and volume average) give the same results.
The solution of the homogenization problem allows not only to establish the macroscopic response of the composite to the external loading but to connect its overall characteristics with the details of the micro-structure (the components properties, sizes and shapes of inclusions and their spatial distribution in the matrix). The knowledge of such dependence is necessary for the theoretical description of the structures from the composite materials and for designing composite materials with prescribed properties.
Note that matrix composites may be reinforced by regular (periodical) as well as random system of inclusions. In the former case the homogenization problem can be solved with any degree of accuracy if advanced computer techniques are used. But in the case of stochastic composites this problem can be solved only approximately. The main difficulties are in the description of many particles interaction between the randomly placed inclusions.
There are several methods of the homogenization problem solution including the bounding of the effective characteristics with the help of special variational principles and composites with the periodical micro-structures which are used as the

models of the real materials. But we restrict ourselves to the consideration of the so-called self-consistent approaches.

There is a group of methods in theoretical physics known as self-consistent methods, which allow to solve effectively the many-particles problem. As a rule these methods reduce the many-particles problem to a one-particle problem and thus provide the opportunity for the effective solution of the homogenization problem. To show how the ideas of these approaches can be applied to the solution of this problem in the mechanics of composite materials is the primary aim of these lectures.

The success of self-consistent methods strongly depends on the analytical solution of the so-called one-particle problem, i.e. the solution of the elastic problem for a single inclusion in the unbounded elastic medium. In what follows we start with the consideration of this problem in detail.

2 The single inclusion problem

Here the so-called single inclusion (or one-particle) problem is treated. It concerns the determination of the displacement field in an elastic solid, containing an inhomogeneity whose elastic properties differ from those of the surrounding medium (matrix). Having in mind the application to the heterogeneous medium, the main interest for us are the strain and stress fields within the inhomogeneity. These fields turn out to be constant for the ellipsoidal inclusion. The most important cases – sphere, disk or fiber are considered and listed.

2.1 The Green's function for the elastic displacement

We consider the unbounded elastic medium that is characterized by the elastic moduli tensor C^0_{ijkl}. Let us suppose that the body forces with density $q_i(x)$ are given. We assume that these forces are distributed in a closed region V and equal to zero outside this region. Then the equation of elastic equilibrium takes the form

$$\partial_j C^0_{ijkl} \partial_l u_k(x) = q_i(x) V(x), \quad \partial_j = \frac{\partial}{\partial x_j} \tag{2.1}$$

where $V(x)$ is the characteristic function of the region V. This function is determined as

$$V(x) = \begin{cases} 1 & x \in V \\ 0 & x \notin V \end{cases} \tag{2.2}$$

Let $G_{ij}(x)$ is the Green's function for displacement in the matrix, i.e.

$$\partial_j C^0_{ijkl} \partial_l G_{kj}(x) + \delta(x)\delta_{ij} = 0 \tag{2.3}$$

where $\delta(x)$ is 3-D Dirac's function. If the elastic displacements $u_i(x)$ are equal to zero at infinity, the Green's function allows to recast Eq. (2.1) as

$$u_i(x) = \int_V G_{ij}(x - x')q_j(x')dx' \tag{2.4}$$

To find the explicit expression for the tensor $G(x)$ in the general case of an arbitrary anisotropic medium is impossible. So we restrict ourselves by the isotropic medium, for which it is easy enough to obtain the explicit expression for the tensor $G(x)$.
One of the possible ways to do it is using the Fourier transformation. We have

$$G_{ij}(k) = \int G_{ij}(x)e^{-ik\cdot x}dx \tag{2.5}$$

Here the same notation is used for the Fourier transformation of the function with argument k that is the parameter of the transformation. The inverse Fourier transformation is determined by the expression

$$G_{ij}(x) = \frac{1}{(2\pi)^3} \int G_{ij}(k)e^{ik\cdot x}dk \tag{2.6}$$

We have further

$$ik_m G_{ij}(k) = \int \partial_m G_{ij}(x)e^{-ik\cdot x}dx, \quad 1 = \int \delta(x)e^{-ik\cdot x}dx \tag{2.7}$$

Using these relations we obtain the algebraic equation for the function $G_{ij}(k)$

$$C^0_{ijkl}k_j k_l G_{km}(k) = \delta_{im} \tag{2.8}$$

For an isotropic medium

$$C^0_{ijkl} = \lambda_0 \delta_{ij}\delta_{kl} + \mu_0(\delta_{ik}\delta_{lj} + \delta_{il}\delta_{kj}) \tag{2.9}$$

the Eq. (2.8) takes the form

$$(\lambda_0 + \mu_0)k_i k_p G_{pq}(k) + \mu_0 \delta_{ip} k^2 G_{pq}(k) = \delta_{iq} \tag{2.10}$$

The multiplying this equation on k_i allows us to find

$$k_\mu G_{\mu\rho}(k) = \frac{1}{\lambda_0 + 2\mu_0} \frac{k_\rho}{k^2} \tag{2.11}$$

Substituting (2.11) into (2.10) we obtain the final expression for $G_{ij}(k)$

$$G_{ij}(k) = \frac{1}{\mu_0}\left(\frac{1}{k^2}\delta_{ij} - \kappa_0 \frac{k_i k_j}{k^4}\right), \quad \kappa_0 = \frac{\lambda_0 + \mu_0}{\lambda_0 + 2\mu_0} \tag{2.12}$$

It follows now from (2.6) and (2.12)

$$G_{ij}(x) = \frac{1}{(2\pi)^3 \mu_0}\left(\delta_{ij}\int \frac{1}{k^2} e^{ik\cdot x} dk - \kappa_0 \int \frac{k_i k_j}{k^4} e^{ik\cdot x} dk\right) \tag{2.13}$$

To calculate these integrals we utilize the known relations

$$\Delta^2 r(x) \equiv \partial_i \partial_j \partial_i \partial_j r(x) = -8\pi\delta(x), \quad r(x) = \sqrt{x_1^2 + x_2^2 + x_3^2} \tag{2.14}$$

Taking into account that under Fourier transformation $\partial_i \to ik_i$, we find

$$k^4 r(k) = -8\pi \tag{2.15}$$

It follows from here

$$r(x) = -\frac{1}{\pi^2}\int \frac{1}{k^4} e^{ik\cdot x} dk \tag{2.16}$$

and

$$\frac{\partial^2 r}{\partial x_i \partial x_j} = \frac{1}{\pi^2}\int \frac{k_i k_j}{k^4} e^{ik\cdot x} dk, \quad \delta_{ij}\frac{\partial^2 r}{\partial x_i \partial x_j} \equiv \frac{2}{r} = \frac{1}{\pi^2}\int \frac{1}{k^2} e^{ik\cdot x} dk \tag{2.17}$$

With the help of Eq. (2.17) the expression (2.12) then becomes

$$G_{ij}(x) = \frac{1}{8\pi\mu_0}\left(\frac{2}{r}\delta_{ij} - \kappa_0 \frac{\partial^2 r}{\partial x_i \partial x_j}\right) \tag{2.18}$$

It can be shown that in the general case the Green's function can be represented in the form

$$G_{ij}(x) = \frac{1}{r} g_{ij}(n) \tag{2.19}$$

where $g(n)$ is the function defined on the unit sphere (n is the unit normal to it).

2.2 The isolated inhomogeneity in an elastic medium

We consider now the unbounded elastic medium with the tensor of elastic moduli C^0_{ijkl} containing the closed region V (inhomogeneity) with other elastic properties characterized by the tensor C_{ijkl} (Fig. 3). When the body forces are absent the equation of equilibrium is

$$\partial_j C_{ijkl}(x)\partial_l u_k(x) = 0 \tag{2.20}$$

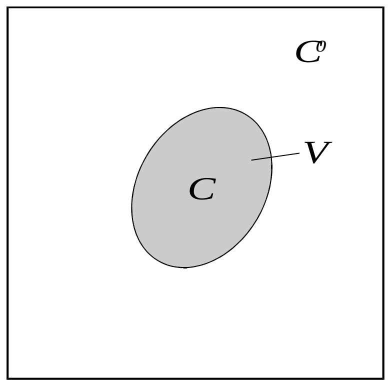

Fig. 3

The tensor $C(x)$ as a function of the coordinates can be represented as

$$C_{ijkl}(x) = C^0_{ijkl} + C^1_{ijkl}(x) \tag{2.21}$$

where we have denoted

$$C^1_{ijkl}(x) = C^1_{ijkl}V(x), \quad C^1 = C - C^0 \tag{2.22}$$

and $V(x)$ is the characteristic function of the region V.
The representation (2.21) allows us to rewrite Eq. (2.20) in the form

$$\partial_j C^0_{ijkl}\partial_l u_k(x) = -\partial_j C^1_{ijkl}(x)\varepsilon_{kl}(x) \tag{2.23}$$

where

$$\varepsilon_{kl}(x) = \frac{1}{2}\left(u_{k,l}(x) + u_{l,k}(x)\right)$$

is the strain tensor. We can consider the right-hand side of Eq. (2.23) as a distribution of some body forces in the homogeneous medium with the elastic moduli tensor C^0. Using the Green's function determined above for such medium, we can write

$$u_i(x) = u_i^0(x) + \int\limits_V G_{ik,l}(x - x')C^1_{klmn}\varepsilon_{mn}(x')dx' \tag{2.24}$$

where Gauss' theorem has been used.
Here $u^0(x)$ is the external elastic field which would arise in the main medium if there was no inhomogeneity and for a specific condition at infinity. In other words $u_i^0(x)$ satisfies the equation of equilibrium

$$\partial_j C^0_{ijkl}\partial_l u_k^0(x) = 0 \tag{2.25}$$

under the given condition at infinity.
Taking derivatives with respect to the coordinates on both sides of Eq. (2.24) we obtain the basic integral equation that is equivalent to the initial differential equation of the problem (2.20)

$$\varepsilon_{ij}(x) = \varepsilon_{ij}^0(x) + \int\limits_V P_{ijkl}(x - x')C^1_{klmn}\varepsilon_{mn}(x')dx' \tag{2.26}$$

where we have denoted

$$P_{ijkl}(x) = \left.\frac{\partial^2 G_{ik}(x)}{\partial x_j \partial x_l}\right|_{(ij)(kl)}, \quad \varepsilon_{ij}^0(x) = u^0_{(i,j)}(x) \tag{2.27}$$

The function $P_{ijkl}(x)$ is called the Green's function for the strain in an elastic medium. Let us note that the function $G(x)$ has a singularity r^{-1}, when $r \to 0$. Hence, the second derivative of the Green's function has the singularity r^{-3} and the integral in (2.26) makes no sense from the ordinary point of view. To give sense to this integral the kernel $P(x)$ in the integral equation (2.26) has to be considered as a generalized function. The construction of algorithms for the formally diverging integral calculation is named regularization. There is such a regularization in this case but we do not need it in what follows and therefore we do not stop on the details.
An equation analogous to (2.26) can be obtained for the stresses σ_{ij}. Multiplying both sides of Eq. (2.26) by the tensor C^0 and taking into account that $\varepsilon = S\sigma$, where $S = C^{-1}$ is the tensor of elastic compliances, we obtain

$$C^0_{ijkl}S_{klmn}\sigma_{mn}(x) = \sigma_{ij}^0(x) + \int\limits_V C^0_{ijrs}P_{rskl}(x - x')C^1_{klmn}S_{mnpq}\sigma_{pq}(x')dx'$$

With the help of the obvious relations

$$C^0_{ijkl}S_{klmn} = C^0_{ijkl}(S^0_{klmn} + S^1_{klmn}) = I_{ijmn} + C^0_{ijkl}S^1_{klmn},$$

$$C^1_{ijkl}S_{klmn} = (C_{ijkl} - C^0_{ijkl})S_{klmn} = I_{ijmn} - C^0_{ijkl}S_{klmn} = -C^0_{ijkl}S^1_{klmn}$$

where we have denoted

$$S^1 = S - S^0, \quad I = (I_{ijkl}) = \frac{1}{2}(\delta_{ik}\delta_{lj} + \delta_{il}\delta_{kj})$$

we obtain finally

$$\sigma_{ij}(x) = \sigma^0_{ij}(x) + \int\limits_V Q_{ijkl}(x - x')S^1_{klmn}\sigma_{mn}(x')dx' \tag{2.28}$$

$$\sigma^0_{ij}(x) = C^0_{ijkl}\varepsilon^0_{kl}, \quad Q_{ijkl}(x) = -C^0_{ijmn}(I_{mnkl}\delta(x) + P_{mnrs}(x)C^0_{rskl})$$

The tensor $Q(x)$ is named the Green's function for the stress field.
If the point x is in V, the equations (2.26) and (2.28) can be considered as integral equations for the fields $\varepsilon(x)$ and $\sigma(x)$ inside of the region V, when the external fields $\varepsilon^0(x)$ and $\sigma^0(x)$ are given. If the fields inside the inclusion are known, the stress and strain fields outside of it can be determined uniquely.
We can consider the elastic problem for a medium with inhomogeneity in the stresses

$$\partial_j\sigma_{ij}(x) = 0, \quad \mathrm{Curl}_{ijkl}S_{klmn}(x)\sigma_{mn}(x) = 0 \tag{2.29}$$

where

$$\mathrm{Curl}_{ijkl} \equiv \epsilon_{ipk}\epsilon_{jql}\partial_p\partial_q$$

where ϵ_{ipq} is the unit antisymmetric tensor that is determined by the expressions

$$\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1, \quad \epsilon_{132} = \epsilon_{213} = \epsilon_{321} = -1$$

$$\epsilon_{ipq} = 0 \quad \text{in all other cases.}$$

Representing the tensor $S(x)$ in the form

$$S(x) = S^0 + S^1(x), \quad S^1(x) = S^1V(x), \quad S^1 = S - S^0 \tag{2.30}$$

we can rewrite Eq. (2.29) as

$$\mathrm{Curl}_{ijkl}S^0_{klmn}\sigma_{mn}(x) = -\mathrm{Curl}_{ijkl}S^1_{klmn}(x)\sigma_{mn}(x) \tag{2.31}$$

The right-hand side of this expression can be considered as a source of internal stresses (the incompatibility tensor). If we introduce the Green's function $Z_{ijkl}(x)$ for the internal stresses that satisfies the equation

$$\mathrm{Curl}_{ijpq}Z_{pqkl}(x) = -I_{ijkl}\delta(x) \tag{2.32}$$

we can write using Stokes' theorem

$$\sigma_{ij}(x) = \sigma^0_{ij}(x) + \int_V \mathrm{Rot}_{ijpq}Z_{pqrs}(x-x')S^1_{rskl}\sigma_{kl}(x')dx' \tag{2.33}$$

Comparison of this equation with (2.28) leads to relation

$$\mathrm{Curl}_{ijpq}Z_{pqrs}(x) = Q_{ijrs}(x) \tag{2.34}$$

Hence, when the incompatibility tensor is the Curl of some function, the Green's function for the internal stresses can be expressed via the Green's tensor for the displacements $G(x)$.

Let us consider now the thermoelastic problem for the medium with an inhomogeneity, i.e. for the medium with the following governing relation

$$\varepsilon_{ij}(x) = S_{ijkl}(x)\sigma_{kl}(x) + \alpha_{ij}(x)T(x) \tag{2.35}$$

where $\alpha(x)$ is the tensor of thermal expansion coefficients, $T(x)$ is the temperature field.

We can represent, as previously, the tensors $S(x)$ and $\alpha(x)$ in the form

$$S(x) = S^0 + S^1(x), \quad \alpha(x) = \alpha^0 + \alpha^1(x), \quad \alpha^1 = \alpha - \alpha^0 \tag{2.36}$$

and assume that the medium with inclusion is under the action of given forces at infinity and in a known temperature field. The stress tensor in such medium satisfies the equation

$$\mathrm{div}\sigma(x) = 0, \quad \mathrm{Curl}S^0\sigma(x) = -\mathrm{Curl}\left[S^1(x)\sigma(x) + \alpha(x)T(x)\right] \tag{2.37}$$

If we assume that the temperature field is uniform, we obtain in analogy with (2.33)

$$\sigma(x) = \sigma^0(x) + \int_V Q(x-x')\left[S^1\sigma(x') + \alpha^1 T\right]dx' \tag{2.38}$$

When $T = 0$ this equation coincides with (2.28). If the external stress field is absent ($\sigma^0(x) = 0, \quad T \neq 0$) the equation (2.38) describes the thermal stress distribution in the medium with an inhomogeneity.

Let us note an important property of integral equations (2.26) and (2.28). If we represent the fields $\varepsilon(x)$ and $\sigma(x)$ in the medium with inhomogeneity in the form

$$\varepsilon(x) = \varepsilon^0(x) + \varepsilon^1(x), \quad \sigma(x) = \sigma^0(x) + \sigma^1(x) \tag{2.38}$$

where $\varepsilon^1(x)$ and $\sigma^1(x)$ are the disturbances of the external fields due to the presence of the inhomogeneity, we can rewrite Eq. (2.26) and (2.28) as

$$\varepsilon^1(x) - \int_V P(x-x')C^1\varepsilon^1(x')dx' = \int_V P(x-x')C^1\varepsilon^0(x')dx' \tag{2.39}$$

$$\sigma^1(x) - \int_V Q(x-x')S^1\sigma^1(x')dx' = \int_V Q(x-x')S^1\sigma^0(x')dx' \tag{2.40}$$

It follows from here that the fields $\varepsilon^1(x)$ and $\sigma^1(x)$ depend only on the values of the external fields $\varepsilon^0(x)$ and $\sigma^0(x)$ in the region occupied by the inclusion.

2.3 The conditions on the interface of two media

Before we start to solve the one-particle problem for the inclusion of a particular shape, let us consider the problem of stress and strain discontinuity on the interface of two media. Let two elastic media with tensors of elastic moduli C^0 and C be ideally joined (Fig. 4). It means that on the (sufficiently smooth) interface Ω between them the usual continuity conditions of displacement and of the normal stress vector

$$u_i^0 = u_i, \quad n_j\sigma_{ij}^0 = n_j\sigma_{ij} \tag{2.41}$$

are satisfied. Let us find an expression for the discontinuity of the stress and strain at a point on the interface Ω .

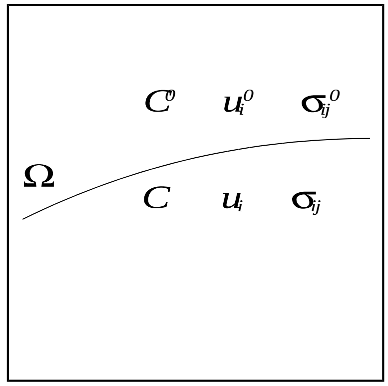

Fig. 4

From the continuity of u on Ω, follows the continuity of the tangential component of the tensor ∇u on Ω. Introducing the projection operators onto the normal and onto the tangent plane

$$\pi_{ij} = n_i n_j, \quad \theta_{ij} = \delta_{ij} - n_i n_j$$

let us write down these conditions in the form

$$\theta_{ik}\partial_k u_j^0 = \theta_{ik}\partial_k u_j \tag{2.42}$$

To transform the second condition (2.41) we have

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} = C_{ijkl}\partial_k u_l$$

due to symmetry of the tensor of elastic moduli. Decomposing ∇u^0 into the sum of normal and tangent components, we obtain

$$n_j C_{ijkl}^0 \pi_{kp}\partial_p u_l^0 + n_j C_{ijkl}^0 \theta_{kp}\partial_p u_l^0 = n_j C_{ijkl}\partial_k u_l \tag{2.43}$$

Using (2.41), we obtain the equation for $n \cdot \nabla u^0$

$$L_{ik}^0(n) n_p \partial_p u_k^0 = L_{ik}^0(n) n_p \partial_p u_k + n_j C_{ijkl}^1 \partial_l u_k \tag{2.44}$$

where

$$L_{ik}^0(n) = C_{ijkl}^0 n_j n_l, \quad C_{ijkl}^1 = C_{ijkl} - C_{ijkl}^0 \tag{2.45}$$

Applying to both sides of (2.44) a matrix $G^0(n)$, which is inverse to $L^0(n)$, we solve (2.44) with respect to $n \cdot \nabla u^0$. Then, multiplying (tensorially) the obtained result by n, we find

$$\pi_{ik}\partial_k u_j^0 = \pi_{ik}\partial_k u_j + n_i G_{jk}^0(n) n_l C_{klmn}^1 \partial_m u_n \tag{2.46}$$

Adding this to (2.42) and symmetrizing with respect to the indices ij , we have

$$[\varepsilon_{ij}] = \varepsilon_{ij} - \varepsilon^0_{ij} = -P^0_{ijkl}(n)C^1_{klmn}\varepsilon_{mn} = +P_{ijkl}(n)C^1_{klmn}\varepsilon^0_{mn} \tag{2.47}$$

where

$$P^0_{ijkl}(n) = \left[n_i G^0_{jk}(n) n_l\right]_{(ij)(kl)}, \quad P_{ijkl}(n) = \left[n_i G_{jk}(n) n_l\right]_{(ij)(kl)}. \tag{2.48}$$

Using direct notation (without the indices) one can rewrite (2.46) in the form

$$\varepsilon = (I + P(n)C^1)\varepsilon_0, \quad \varepsilon_0 = (I + P^0(n)C^1)\varepsilon \tag{2.49}$$

From here it follows in particular that

$$P - P^0 + PC^1P^0 = 0 \tag{2.50}$$

It is not difficult to obtain formulae for the discontinuity of the stress from the formulae for the discontinuity of the strain. We present these formulae in a form which is completely symmetric to (2.46)-(2.49)

$$[\sigma] = -Q^0S^1\sigma = +QS^1\sigma_0$$

$$\sigma = (I + QS^1)\sigma_0, \quad \sigma_0 = (I + Q^0S^1)\sigma \tag{2.51}$$

$$Q - Q^0 + QS^1Q^0 = 0$$

Here

$$Q(n) = C - CP(n)C, \quad S^1 = S - S^0$$

and similarly for $Q^0(n)$.

One has to notice a remarkable correspondence between local coefficients in the formulae for the discontinuity of the stress and strain and the Green's function introduce above. This correspondence was reflected above in the adopted notations. In fact the matrix $G^0(n)$ is, by definition, the inverse of the matrix $L^0(n)$, which is given by (2.45). Comparison with the formulae of Section 3.1 shows that $G^0(n)$ coincides with the value of the Green's function $G^0(k)$ of the homogeneous medium on the unit sphere in k-space. As regards $P^0(n)$ and $Q^0(n)$ they coincide exactly with the Green's functions of the homogeneous medium for strain and stress in k-representation.

2.4 Ellipsoidal inhomogeneity

Discovered by J. Eshelby in the middle of this century, the remarkable property of the inhomogeneity of ellipsoidal shape played, perhaps, the main role in the following development of the composite material theory. There are several proofs of Eshelby's theorem. We choose here maybe not the shortest but the simplest and most transparent of them.

As was mentioned above, the Green's function for displacements can be represented in the following general form

$$G(\mathbf{x}) = \frac{1}{r} G^*(\mathbf{a}^r), \quad \mathbf{a}^r = \frac{\mathbf{x}}{r}, \quad r = |\mathbf{x}| \tag{2.52}$$

where $G^*(\mathbf{a}^r)$ is the function defined on the unit sphere. It can be shown that

$$G^*(\mathbf{a}^r) = G^*(-\mathbf{a}^r) \tag{2.53}$$

i.e. function $G^*(\mathbf{a}^r)$ is symmetric with respect to the center of the unit sphere. Let us introduce the local tangential basis of the spherical system of coordinates (Fig. 5)

$$\mathbf{a}^r = \frac{\partial \mathbf{x}}{\partial r}, \quad \mathbf{a}^\varphi = \frac{1}{r \sin\theta} \frac{\partial \mathbf{x}}{\partial \varphi}, \quad \mathbf{a}^\theta = \frac{1}{r} \frac{\partial \mathbf{x}}{\partial \theta} \tag{2.54}$$

In this basis we have

$$\nabla = \mathbf{a}^r \frac{\partial}{\partial r} + \nabla^*, \quad \nabla^* = \mathbf{a}^\varphi \frac{1}{\sin\theta} \frac{\partial}{\partial \varphi} + \mathbf{a}^\theta \frac{\partial}{\partial \theta} \tag{2.55}$$

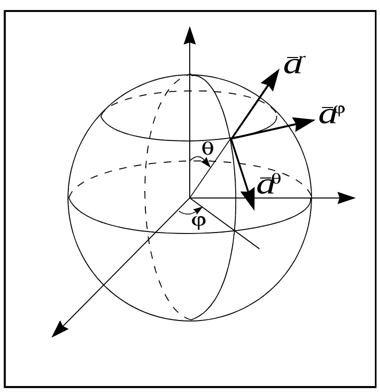

Fig. 5

It follows from (2.54) and (2.55) that

$$\nabla G(\mathbf{x}) = \frac{1}{r^2} G^1, \quad G^1 = \nabla^* G^* - G^* \mathbf{a}^r \tag{2.56}$$

where the function G^1 is also determined on the surface of the unit sphere. By contrast to G^* the function G^1 is antisymmetric with respect to the center of this sphere, i.e.

$$G^1(-\mathbf{a}^r) = -G^1(\mathbf{a}^r) \tag{2.57}$$

This fact is essential in the proof of Eshelby's theorem, which can be formulated in the following way:
if the external field $\varepsilon^0(x)$ is uniform in the region V , then the field ε inside this region also uniform.
To prove this statement it is enough to show, as follows from equation (2.24), that the integral

$$\int\limits_V \nabla G(x-x')dx' \quad (x \in V) \tag{2.58}$$

is a linear function of the point x.
It has to be noted that the integrand in (2.58) is the function with singularity r^{-2} but this integral exists in the ordinary sense because in spherical coordinates $dv = r^2 dr d\varphi \sin\theta d\theta$.

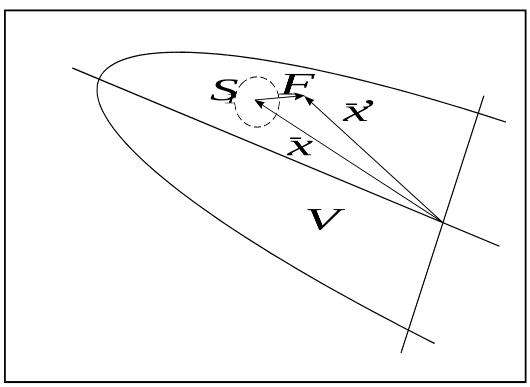

Fig. 6

Let us assume that the origin of the vector $\mathbf{x}$ is fixed in V (Fig. 6). Denoting $\mathbf{r} = \mathbf{x}' - \mathbf{x}$ we can write

$$\int\limits_V \nabla G(x-x')dx' = -\int\limits_0^{r_s} dr \int\limits_{S_1} (\nabla^* G^* - G^* \mathbf{a}^r) dS_1 \tag{2.59}$$

where r_s denotes the value of the vector $\mathbf{r}$ when its end is placed on the surface of the ellipsoid, S^1 is the unit sphere with the center in the point x. Let $\mathbf{T}$ be the second rank tensor that is characterized by the ellipsoidal form

$$\mathbf{T} = \frac{\mathbf{a}^1 \otimes \mathbf{a}^1}{a_1^2} + \frac{\mathbf{a}^2 \otimes \mathbf{a}^2}{a_2^2} + \frac{\mathbf{a}^3 \otimes \mathbf{a}^3}{a_3^2} \tag{2.60}$$

where $\mathbf{a}^1, \mathbf{a}^2, \mathbf{a}^3$ represent the unit vectors of the ellipsoid axes of symmetry, a_1, a_2, a_3 are the semi-axes of the ellipsoid and

$$\mathbf{x}^s \cdot \mathbf{T} \cdot \mathbf{x}^s = 1 \tag{2.61}$$

is the equation of the ellipsoid surface ($\mathbf{x}^s$ is the vector $\mathbf{x}'$ when its end is on this surface). Because of $\mathbf{r} = \mathbf{x}' - \mathbf{x}$ now and $\mathbf{x}^s = \mathbf{r}^s + \mathbf{x}$, it follows from (2.61)

$$(r_s \mathbf{a}^r + \mathbf{x}) \cdot \mathbf{T} \cdot (r_s \mathbf{a}^r + \mathbf{x}) = \mathbf{1}$$

This is a quadratic equation with respect to r_s having only one positive root

$$r_s = (\mathbf{a}^r \cdot \mathbf{T} \cdot \mathbf{a}^r)^{-1} \left[-\mathbf{a}^r \cdot \mathbf{T} \cdot \mathbf{x} + \sqrt{(\mathbf{a}^r \cdot \mathbf{T} \cdot \mathbf{a}^r)^2 - (\mathbf{a}^r \cdot \mathbf{T} \cdot \mathbf{a}^r)(\mathbf{x} \cdot \mathbf{T} \cdot \mathbf{x} - 1)} \right] \tag{2.62}$$

Since $\mathbf{x}^s \cdot \mathbf{T} \cdot \mathbf{x}^s = 1$ we have $\mathbf{x} \cdot \mathbf{T} \cdot \mathbf{x} - 1 \leq 0$, when $\mathbf{x} \in \mathbf{V}$. So the expression under the square root in (2.62) is always positive and r_s is actually non negative. Equation (2.59) can be rewritten in the form

$$\int\limits_V \nabla G(x - x')dx' = -\int\limits_{S_1} (\nabla^* G^* - G^* \mathbf{a}^r) r_s dS_1 \tag{2.63}$$

As follows from (2.62), r_s is also a function that is defined on the unit sphere. This function consists of two parts: the first one is antisymmetric and the second one is symmetric with respect to the center of the sphere. The result of the multiplication of two anti-symmetric functions is a symmetric function. Therefore putting the expression for r_s from (2.62) in the right-hand side of (2.63) leads to a sum of symmetric and anti-symmetric functions. The integral of the latter over the unit sphere equals zero. Thus we finally obtain

$$\int\limits_V \nabla G(x - x')dx' = P \cdot x \quad (x \in V) \tag{2.64}$$

where

$$P = \mathbf{T} \cdot \int\limits_{S_1} (\mathbf{a}^r \cdot \mathbf{T} \cdot \mathbf{a}^r)^{-1} \mathbf{a}^r (\nabla^* G^*(\mathbf{a}^r) - G^*(\mathbf{a}^r)\mathbf{a}^r) dS_1 \tag{2.65}$$

is tensor with constant components.
Hence, it follows from Eq. (2.26) that in the case of an ellipsoidal inhomogeneity and uniform external strain field in V, the strain field inside of the inclusion is determined by the expression

$$\varepsilon = A\varepsilon^0, \quad A = (I - P(a)C^1)^{-1} \tag{2.66}$$

The analogous result can be obtained from equation (2.28) in the case of an external stress field uniform in V:

$$\sigma = B\sigma^0, \quad B = (I + Q(a)S^1)^{-1}, \quad Q(a) = C^0 + C^0 P(a) C^0 \tag{2.67}$$

The tensors A and B that appeared in Eq. (2.66) and (2.67) play a central role in the elementary models of composite media, as will be seen below. That is why we shall specify them now for V of ellipsoidal shape and some particular cases – sphere, fiber or disk. To calculate these expressions it is necessary to know the explicit expressions for the Green's function. For an isotropic medium it was obtained above (see Eq. (2.18)). Avoiding the technical details of integration over the unit sphere we present here only the final results.
For the isotropic solid the tensors $P(a)$ and $Q(a)$ have the symmetry of the ellipsoid and are determined by nine essential components. In the coordinate system connected to the ellipsoid axes we can write

$$P_{1111} = -\frac{\kappa_0}{8\pi\mu_0}\left[3J_{11} + (1-4\nu_0)J_1\right], \quad P_{1122} = -\frac{\kappa_0}{8\pi\mu_0}(J_{21} - J_1),$$

$$P_{1212} = \frac{\kappa_0}{8\pi\mu_0}\left[J_{21} + J_{12} + (1-2\nu_0)(J_1 + J_2)\right] \tag{2.68}$$

$$Q_{1111} = 4\mu_0\kappa_0\left[1 - \frac{1}{8\pi}(3J_{11} + J_1)\right],$$

$$Q_{1122} = 4\mu_0\kappa_0\left\{\nu_0 - \frac{1}{16\pi}\left[J_{21} + J_{12} - (1-4\nu_0)(J_1 + J_2)\right]\right\}$$

$$Q_{1212} = 4\mu_0\kappa_0\left\{\frac{1-\nu_0}{2} - \frac{1}{16\pi}\left[J_{21} + J_{12} - (1-2\nu_0)(J_1 + J_2)\right]\right\}$$

where, as earlier,

$$\kappa_0 = \frac{\lambda_0 + \mu_0}{\lambda_0 + 2\mu_0} = \frac{1}{2(1-\nu_0)}$$

(ν_0 is the Poisson coefficient of the matrix). The quantities

$$J_p = \frac{3}{2} v \int_0^\infty \frac{du}{(a_p^2+u)\Delta(u)}, \quad J_{pq} = \frac{3}{2} v a_p^2 \int_0^\infty \frac{du}{(a_p^2+u)(a_q^2+u)\Delta(u)}, \tag{2.69}$$

$$\Delta(u) = \sqrt{(a_1^2+u)(a_2^2+u)(a_3^2+u)}, \quad v = \frac{4}{3}\pi a_1 a_2 a_3, \quad p, q = 1, 2$$

are expressed in terms of elliptic integrals. The remaining six tensor components are obtained by a cyclic replacement of the indices 1,2,3.

We consider now the case, when the inclusion has spheroidal shape with semi-axes $a_1 = a_2 = a, \quad a_3$ (x_3 - axis coinciding with the axis of revolution). In this case tensors P and Q become transversely isotropic and the formulae for their components do not contain elliptic integrals.

For presentation of the tensors having the hexagonal (or transversely isotropic) symmetry it is very convenient to use the special tensor basis made up of Kronecker delta or projector on the x_1x_2 - plane $\theta_{ij} = \delta_{ij} - m_i m_j$ and the unit vector of the x_3 - axis, m_i

$$T^1_{ijkl} = \theta_{i(k}\theta_{l)j}, \quad T^2_{ijkl} = \theta_{ij}\theta_{kl}, \quad T^3_{ijkl} = \theta_{ij} m_k m_l, \tag{2.70}$$

$$T^4_{ijkl} = m_i m_j \theta_{kl}, \quad T^5_{ijkl} = m_{(i}\theta_{j)(k} m_{l)}, \quad T^6_{ijkl} = m_i m_j m_k m_l$$

Let the inclusion be an oblate spheroid ($a > a_3$). Then tensor P takes the form in the basis (2.70):

$$P_{ijkl} = -\left[P_1 T^2_{ijkl} + P_2\left(T^1_{ijkl} - \frac{1}{2}T^2_{ijkl}\right) + P_3(T^3_{ijkl} + T^4_{ijkl}) + P_5 T^5_{ijkl} + P_6 T^6_{ijkl}\right] \tag{2.71}$$

$$P_1 = \frac{1}{2\mu_0}\left[(1-\kappa_0) f_0 + f_1\right], \quad P_2 = \frac{1}{2\mu_0}\left[(2-\kappa_0) f_0 + f_1\right],$$

$$P_3 = -\frac{1}{\mu_0} f_1, \quad P_5 = \frac{1}{\mu_0}\left(1 - f_0 - 4f_1\right), \quad P_6 = \frac{1}{\mu_0}\left[(1-\kappa_0)(1-2f_0) + 2f_1\right]$$

$$f_0 = \frac{1-g}{2(1-\gamma^2)}, \quad f_1 = \frac{\kappa_0}{4(1-\gamma^2)^2}\left[(2+\gamma^2)g - 3\gamma^2\right], \tag{2.72}$$

$$g = \frac{\gamma^2}{\sqrt{\gamma^2-1}} \arctan\sqrt{\gamma^2-1}, \quad \gamma = \frac{a}{a_3} > 1$$

The expressions for the components of the tensor Q in the same basis can be obtained from (2.67) with the help of the following rule of multiplication (convolution over two indices) of two tensors represented in the basis (2.70): if two tensors A and B are represented in T-basis, the multiplication-contraction of these tensors over two indices is determined by:

$$A_{ijkl}B_{klmn} = (2A_1B_1 + A_3B_4)T^2_{ijkl} + A_2B_2\left(T^1_{ijkl} - \frac{1}{2}T^2_{ijkl}\right) + (2A_1B_3 + \tag{2.73}$$

$$+A_3B_6)T^3_{ijkl} + +(2A_4B_1 + A_6B_4)T^4_{ijkl} + \frac{1}{2}A_5B_5T^5_{ijkl} + (A_6B_6 + 2A_4B_3)T^6_{ijkl}$$

We have to add to this the following important relation concerning the tensor representation in T-basis. If A is a tensor presented in this basis, then the inverse tensor A^{-1} is determined by the expression

$$A^{-1} = \frac{A_6}{2\Delta}T^2 + \frac{1}{A_2}\left(T^1 - \frac{1}{2}T^2\right) - \frac{A_3}{\Delta}T^3 - \frac{A_4}{\Delta}T^4 + \frac{4}{A_5}T^5 + \frac{2A_1}{\Delta}T^6 \tag{2.74}$$

$$\Delta = 2(A_1A_6 - A_3A_4)$$

With the help of (2.73) we have

$$Q = Q_1T^2 + Q_2\left(T^1 - \frac{1}{2}T^2\right) + Q_3(T^3 + T^4) + Q_5T^5 + Q_6T^6 \tag{2.75}$$

$$Q_1 = \mu_0\left[4\kappa_0 - 1 - 2(3\kappa_0 - 1)f_0 - 2f_1\right]$$

$$Q_2 = 2\mu_0\left[1 - (2-\kappa_0)f_0 - f_1\right], \quad Q_3 = 2\mu_0\left[(2\kappa_0 - 1)f_0 + 2f_1\right]$$

$$Q_5 = 4\mu_0(f_0 + 4f_1), \quad Q_6 = 4\mu_0\left[(1 + 2\kappa_0)f_0 - 2f_1\right]$$

If $\gamma \gg 1$ (flat disk), then with precision of order γ^{-1} the functions f_0 and f_1 become $f_0 \to \pi/4\gamma, \quad f_1 \to \kappa_0\pi/8\gamma$ and

$$Q_1 = \mu_0(4\kappa_0 - 1) + \frac{\pi\mu_0}{4\gamma}(7\kappa_0 - 2), \quad Q_2 = 2\mu_0 + \frac{\pi\mu_0}{4\gamma}(4 - \kappa_0), \tag{2.76}$$

$$Q_3 = \frac{\pi\mu_0}{2\gamma}(3\kappa_0 - 1), \quad Q_5 = \frac{\pi\mu_0}{\gamma}(1 + 2\kappa_0), \quad Q_6 = \frac{\pi\mu_0}{\gamma}(1 + \kappa_0)$$

In the limit $\gamma \to \infty$ we obtain

$$Q = 2\mu_0\left[T^1 + (2\kappa_0 - 1)T^2\right] \tag{2.77}$$

If the inclusion is a prolate spheroid ($a < a_3, \quad \gamma = a/a_3, \quad \gamma < 1$) then the tensors P and Q are determined by the same formulae presented above in which the function $g(\gamma)$ has to be replaced by

$$g(\gamma) = \frac{\gamma^2}{2\sqrt{1-\gamma^2}} \ln \frac{1 + \sqrt{1-\gamma^2}}{1 - \sqrt{1-\gamma^2}} \tag{2.78}$$

The inclusion in the form of a continuous fiber presents great interest for applications. The very prolate spheroid ($\gamma \ll 1$) can serve as a model of such fiber. The expressions for the tensors P and Q have as earlier, the form (2.71), (2.75), where in the functions f_0 and f_1 have to be kept only the main terms of the series expansion with respect to the small parameter γ

$$f_0 = \frac{1}{2}\left(1 + \gamma^2 - \gamma^2 \ln\frac{2}{\gamma}\right), \quad f_1 = \frac{\kappa_0}{4}\left(2\gamma^2 \ln\frac{2}{\gamma} - 3\gamma^2\right) \tag{2.79}$$

If the parameter γ tends to zero, we come to the expressions for tensors P and Q in the case of a continuous fiber

$$P = -\frac{1}{4\mu_0}\left[(1 - \kappa_0)T^2 + (2 - \kappa_0)\left(T^1 - \frac{1}{2}T^2\right) + 2T^5\right] \tag{2.80}$$

$$Q = \mu_0\left[\kappa_0 T^2 + \kappa_0\left(T^1 - \frac{1}{2}T^2\right) + (2\kappa_0 - 1)(T^3 + T^4) + 2T^5 + 4\kappa_0 T^6\right]$$

In the case of an inclusion in the form of a continuous fiber it is possible to obtain the solution even for the transversely isotropic matrix and fiber under the condition that the common axis of symmetry of both materials is directed along the fiber axis m. Using the tensor basis introduced above we can represent the tensor C^0 in the form:

$$C^0 = k_0 T^2 + 2m_0 \left(T^1 - \frac{1}{2} T^2 \right) + l_0 (T^3 + T^4) + 4\mu_0 T^5 + n_0 T^6 \tag{2.81}$$

Here k_0 is the strain bulk modulus for lateral dilatation without longitudinal extension, m_0 is the rigidity modulus for shearing in any transverse direction, μ_0 is the longitudinal shear modulus, n_0 is the modulus of longitudinal uniaxial extension and l_0 is the associated cross modulus. The same relation with the omission of the subscript 0 can be written for the tensor of elastic moduli of the fiber.

In the considered case of a constant external strain field ε_0 we obtain the following expressions for tensors P and A:

$$P = \frac{1}{4(k_0 + m_0)} T^2 + \frac{k_0 + 2m_0}{4m_0(k_0 + m_0)} \left(T^1 - \frac{1}{2} T^2 \right) + \frac{1}{2\mu_0} T^5 \tag{2.82}$$

$$A = \frac{1}{2} \left(1 + \frac{k_1}{k_0 + m_0} \right)^{-1} T^2 + \left[1 + \frac{m_1(k_0 + 2m_0)}{2(k_0 + m_0)} \right]^{-1} \left(T^1 - \frac{1}{2} T^2 \right) -$$

$$- \frac{l_1}{2(k_0 + m_0)} \left(1 + \frac{k_1}{k_0 + m_0} \right)^{-1} T^3 + \left(1 + \frac{\mu_1}{2\mu_0} \right)^{-1} T^5 + T^6$$

where $k_1 = k - k_0$, $m_1 = m - m_0$, $\mu_1 = \mu - \mu_0$, $l_1 = l - l_0$ are the differences between the elastic moduli of inclusion and the matrix.

Let us return now to the isotropic matrix. If the inclusion has spherical shape, then tensors P and Q do not depend on its radius and become isotropic

$$P_{ijkl} = -\frac{1 - \kappa_0}{9\mu_0} \delta_{ij}\delta_{kl} - \frac{5 - 2\kappa_0}{15\mu_0} \left(I_{ijkl} - \frac{1}{3} \delta_{ij}\delta_{kl} \right) \tag{2.83}$$

$$Q_{ijkl} = \frac{4\mu_0(1 - 4\kappa_0)}{9} \delta_{ij}\delta_{kl} + \frac{2\mu_0(5 + 4\kappa_0)}{15} \left(I_{ijkl} - \frac{1}{3} \delta_{ij}\delta_{kl} \right)$$

Let us note that the tensors $I + PC^1 = A^{-1}$ and $I - QS^1 = B^{-1}$ are invertible and in the case of a hole (pore) and of an absolutely rigid inclusion. In particular, for an arbitrary inclusion we have

$$A = \frac{\lambda_0 + 2\mu_0}{3\lambda_1 + 2\mu_1 + 3(\lambda_0 + 2\mu_0)} \delta_{ij}\delta_{kl} +$$

$$+\frac{15\mu_0(\lambda_0+2\mu_0)}{2\mu_1(3\lambda_0+8\mu_0)+15\mu_0(\lambda_0+2\mu_0)}\left(I_{ijkl}-\frac{1}{3}\delta_{ij}\delta_{kl}\right) \tag{2.84}$$

In the case of a hole ($C^1 = -C^0$) this tensor takes the form

$$A=\frac{\lambda_0+2\mu_0}{4\mu_0}\left[\delta_{ij}\delta_{kl}+\frac{60\mu_0}{9\lambda_0+14\mu_0}\left(I_{ijkl}-\frac{1}{3}\delta_{ij}\delta_{kl}\right)\right] \tag{2.85}$$

If the inclusion is absolutely rigid ($C^1 \to \infty$), then the strain tensor ε inside the inclusion is equal to zero and the tensor B is defined by the expression

$$B=\frac{\lambda_0+2\mu_0}{3(3\lambda_0+2\mu_0)}\left[\delta_{ij}\delta_{kl}+\frac{45(3\lambda_0+2\mu_0)}{2(3\lambda_0+8\mu_0)}\left(I_{ijkl}-\frac{1}{3}\delta_{ij}\delta_{kl}\right)\right] \tag{2.86}$$

Let us consider a medium with an ellipsoidal inclusion in a uniform temperature field T. In this case the stress field σ inside the inclusion is constant and determined by the expression

$$\sigma = B(a)Q(a)\alpha^1 T \tag{2.87}$$

For the inclusion of spherical shape this expression takes the form

$$\sigma_{ij}=\frac{4\mu_0(3\lambda+2\mu)}{4\mu_0+3\lambda+2\mu}\alpha_1 T\delta_{ij} \tag{2.88}$$

Using the expressions obtained above for the discontinuities of the elastic fields on the inclusion/matrix interface

$$\varepsilon^-(x_0)=(I+P^0(n_0)C^1)\varepsilon^+$$

$$\sigma^-(x_0)=(I+Q^0(n_0)S^1)\sigma^+$$

where $x_0 \in \Omega$, $n_0 = n(x_0)$ is the external normal to the interface Ω we can present the limiting outside strain and stress on the interface in the form

$$\varepsilon^-(x_0)=F^\varepsilon(n_0)\varepsilon^0, \quad \sigma^-(x_0)=F^\sigma(n_0)\sigma^0 \tag{2.89}$$

where the tensors

$$F^\varepsilon(n_0)=(I+P^0(n_0)C^1)A, \quad F^\sigma(n_0)=(I+Q^0(n_0)S^1)B \tag{2.90}$$

play the role of tensor coefficients of strain and stress concentration on the surface of the inclusion.

The stress on the interface due to a uniform temperature change are determined by the same expression for $\sigma^-(x_0)$ from (2.89) in which the tensor σ^0 has to be replaced by $Q\alpha_1 T$.

3 Elastic medium with random sets of inhomogeneities

We consider now the composite materials which consist of a homogeneous matrix and a random set of inclusions. As a rule the characteristic sizes of the inclusions and the distances between them are much smaller then the size of the body. Hence, omitting specific edge effects the main information about the structure of the physical fields in a random inhomogeneous medium may be extracted from the solution of the problem for an infinite medium whose properties are described by statistically homogeneous functions in space.

3.1 Integral equations of the problem

Let the matrix tensor moduli be C^0. If the inclusions occupy in the matrix isolated regions V_k with characteristic functions $V_k(x)$, $k = 1, 2, \dots$, then the tensor of elastic moduli of the medium with inclusions can be expressed in the form

$$C(x) = C^0 + C^1(x)V(x), \quad V(x) = \sum_k V_k(x) \tag{3.1}$$

where the function $C^1(x)$ when $x \in V_k$ describes the perturbation of the elastic moduli inside of $k-$th inclusion. In particular, if the elastic moduli are constant in each region V_k then

$$C(x) = C^0 + C^1(x), \quad C^1(x) = \sum_k C_k^1 V_k(x) \tag{3.2}$$

In analogy with Eq. (2.26) and (2.28) the tensors $\varepsilon(x)$ and $\sigma(x)$ in the arbitrary point x of the composite material can be represented in the form

$$\varepsilon(x) = \varepsilon^0(x) + \int P(x-x')C^1(x')\varepsilon(x')dx' \tag{3.3}$$

$$\sigma(x) = \sigma^0(x) + \int Q(x-x')S^1(x')\sigma(x')dx' \tag{3.4}$$

As it has been mentioned above, the kernels $P(x)$ and $Q(x)$ in these equations have the singularity of the order $1/r^3$. Hence, it is important to define the action of the integral operators with these kernels on the constant that formally diverges. Let us consider the following model problem. Let the inhomogeneity with elastic moduli C occupy all space. We suppose there is a constant external stress field σ^0 applied to the medium. The strain and stress fields in such a medium are also constant and have the form

$$\varepsilon = C^{-1}\sigma^0, \quad \sigma = \sigma^0 \tag{3.5}$$

It is not difficult to see that the solutions of equations (3.3) and (3.4) coincide in this case with (3.5) if the following equalities hold:

$$\int P(x-x')dx' = -(C^0)^{-1}, \quad \int Q(x-x')dx' = 0 \tag{3.6}$$

Let now in the problem the constant external field of strain ε^0 be fixed. Then the stress and strain in the medium take the form

$$\varepsilon = \varepsilon^0, \quad \sigma = C\varepsilon^0 \tag{3.7}$$

and

$$\int P(x-x')dx' = 0, \quad \int Q(x-x')dx' = -C^0 \tag{3.8}$$

Hence, it is essential that a unique definition of these operators can be proposed for a given type of external field. Further, in order to be precise, we always should say what kind of external field is supposed to be fixed in the problem.
We introduce now the tensor

$$q(x) = C^1(x)\varepsilon(x) \tag{3.9}$$

Taking into account the relations

$$C^1\varepsilon = C^1 S\sigma = (C - C^0)S\sigma = \tag{3.10}$$

$$= [I - C^0(S^0 + S^1)]\sigma = -C^0 S^1\sigma$$

we can write the equations (3.3) and (3.4) as

$$\varepsilon(x) = \varepsilon^0 + \int P(x-x')q(x')dx' \tag{3.11}$$

$$\sigma(x) = \sigma^0 + \int Q(x-x')S^0 q(x')dx' \tag{3.12}$$

supposing that the fields ε^0 and σ^0 are constant.

Let the set of inclusions be uniformly distributed in space. The homogenization problem that will be considered below is the determination of the average over the ensemble of realizations of the random set of inclusion values of the strain and stress fields in an arbitrary point x of the medium. The connection between the ensemble averages and averages over the representative volume has been discussed above.

As it follows from Eq. (3.11) and (3.12) the mentioned averages are presented in the form

$$\langle\varepsilon(x)\rangle = \varepsilon^0 + \int P(x-x')\,\langle q(x')\rangle\, dx' \tag{3.13}$$

$$\langle\sigma(x)\rangle = \sigma^0 + \int Q(x-x')S^0\,\langle q(x')\rangle\, dx' \tag{3.14}$$

where we have taken into account that $P(x)$ and $Q(x)$ are deterministic functions. For a spatially uniform set of inclusions $\varepsilon(x), \sigma(x)$ and $q(x)$ are homogeneous random functions that possess the ergodic property. Therefore, the average $\langle q\rangle$ is a constant tensor whose value can be found by spatial averaging of a typical fixed realization of the random function $q(x)$:

$$\langle q\rangle = \lim_{W\to\infty}\frac{1}{W}\int\limits_W q(x)dx \tag{3.15}$$

Here W is a region in $3-D$ space occupying in the limit all space.

Because of the linearity of the problem the strain field $\varepsilon(x)$ is connected with the external field ε^0 through the relation

$$\varepsilon(x) = \Lambda(x)\varepsilon^0 \tag{3.16}$$

where $\Lambda(x)$ is the forth rank tensor function that has to be obtained from the solution of the many-particle problem. After substituting the expression for $\varepsilon(x)$ in the formula $q(x)$ in Eq. (3.9) and averaging the result, we obtain

$$\langle q(x)\rangle = pC^{\Lambda}\varepsilon^0, \quad C^{\Lambda} = \left\langle C_v^{\Lambda}\right\rangle, \quad C_v^{\Lambda} = \frac{1}{v}\int\limits_v C^1(x)\Lambda(x)dx \tag{3.17}$$

where p is the volume concentration of the inclusions, $p = \langle V(x)\rangle$, v is the volume of a typical inclusion and the average $\left\langle C_v^{\Lambda}\right\rangle$ is calculated from the ensemble distribution of the random variable C_v^{Λ}.

We assume that the average strain of the inhomogeneous medium is fixed. This strain does not depend on the properties and volume concentration of inclusions and coincides with the external field ε^0 applied to the medium. Let us note that for a finite body, fixing the mean strain means specified displacements on its border Ω. If the displacement field $u^0(x)$ specified on Ω has the form $u_i^0(x) = \varepsilon_{ij}^0 x_j$, where ε_{ij}^0 is the constant symmetric tensor, then the average strain in the inhomogeneous medium is equal to ε^0. It follows from the definition of the average over the volume and Gauss' theorem that:

$$\langle \varepsilon_{ij} \rangle = \frac{1}{2V} \int\limits_V \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) dx = \frac{1}{2V} \int\limits_\Omega (u_i^0 n_j + u_j^0 n_i) d\Omega = \varepsilon_{ij}^0 \tag{3.18}$$

This property holds also for a region occupying all space.
If the average strain is fixed, the effect of the operators P and Q in Eq. (3.12) and (3.13) on the constant tensors $\langle q \rangle$ and $S^0 \langle q \rangle$ is determined by Eq. (3.8). It follows from Eq. (3.13) and (3.14) that the expression for the average stresses in the medium with inclusions has the form

$$\langle \sigma(x) \rangle = C^* \langle \varepsilon(x) \rangle , \quad C^* = C^0 + pC^\Lambda \tag{3.19}$$

where is the tensor of the effective moduli of elasticity of the composite material. Thus the problem of obtaining the effective elastic moduli C^* reduces to the evaluation of the tensor C^Λ determined by Eq. (3.17). This tensor depends on the solution of the many-particle problem through the function $\Lambda(x)$. For the evaluation of C^Λ we will use certain self-consistent schemes. Let us examine these schemes based on the solution of the problem for a single inclusion in a homogeneous medium loaded by a constant external field (i.e. the one-particle problem considered above).

4 Effective medium method

One of the first self-consistent schemes for solving the homogenization problem in mechanics of composites was based on the following hypothesis:
Each inclusion in the composite behaves as an isolated one in a homogeneous medium whose properties coincide with the effective properties of the entire composite. The field acting on this inclusion coincides with the external field ε^0 *applied to the composite*(Fig. 7).

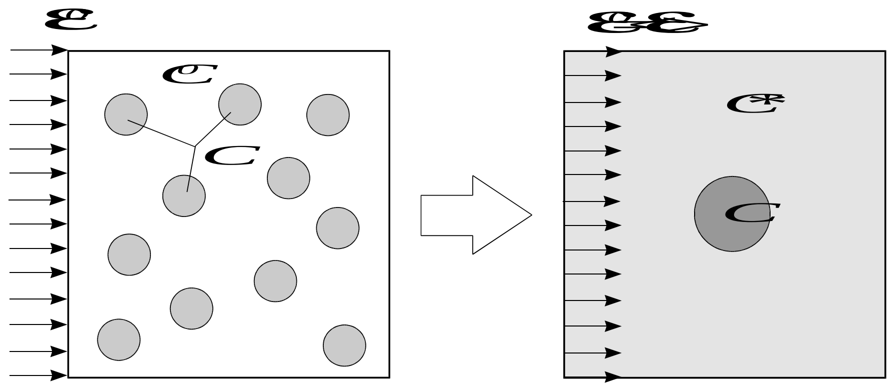

Fig. 7

The self-consistent schemes in which the composite material outside of certain vicinity of each inclusion is replaced by a medium with effective properties are called *the effective medium methods.* On the basis of this hypothesis we can find from the solution of the one-particle problem the dependence of the tensor C^{Λ} from (3.17) on the parameters characterizing the form of the inclusion, its modulus of elasticity C and the effective moduli of elasticity of the composite C^*. In accordance to (3.19) we have

$$C^* = C^0 + p\left\langle C_v^{\Lambda}(C^*, C, \{\widetilde{a}\}\right\rangle \tag{4.1}$$

where $\{\widetilde{a}\} = (\widetilde{a}_1, \widetilde{a}_2, ...)$ is a set of parameters characterizing the shape of the inclusions. This equation can be considered as an equation for the determination of the tensor of the effective elastic medium.

Let us consider some special case when only a single family of isotropic spherical inclusions are uniformly distributed in the matrix. In this case the overall properties of the composite are also isotropic and are characterized by the elastic moduli λ^*, μ^* (or $K^* = \lambda^* + 2\mu^*/3, \quad \mu^*$). Using the solution presented above of the one particle problem for the spherical inclusion in isotropic matrix, we obtain the following expression for the tensor C_v^{Λ}

$$C_v^{\Lambda} = (C - C^0)\left[I + P^*(C - C^*)\right]^{-1} \tag{4.2}$$

where the tensor P^* is determined in Eq. (2.83) in which the parameters λ_0 and μ_0 have to be replaced by λ^* and μ^*. From here and (4.1) we obtain the system of algebraic equations for the determination of the bulk K^* and shear μ^* effective elastic moduli of the composite

$$K^* = K_0 + p(K - K_0)\left[1 + \frac{3(K - K^*)}{3K^* + 4\mu^*}\right]^{-1}$$

$$\mu^* = \mu_0 + p(\mu - \mu_0)\left[1 + \frac{6(\mu - \mu^*)(K^* + 2\mu^*)}{5\mu^*(3K^* + 4\mu^*)}\right]^{-1} \tag{4.3}$$

These equations can be solved at least numerically. To make the solution observable, let us consider the particular case when the inclusions are spherical pores ($K = 0, \mu = 0$). From the first equation one can find

$$K^* = \frac{4(1-p)K_0\mu^*}{3pK_0 + 4\mu^*}$$

and substituting this expression into the second equation we obtain that μ^* satisfies the quadratic equation

$$16\mu^{*2} + \mu^*\left[K_0(3-p) - 4\mu_0(4-5p)\right] - 3\mu_0 K_0(1-2p) = 0 \tag{4.4}$$

In the limit $K_0 \to \infty$, corresponding to the incompressibility of the matrix, these equations imply

$$K^* = \frac{4(1-p)(1-2p)}{p(3-p)}\mu_0, \quad \mu^* = \frac{3(1-2p)}{3-p}\mu_0 \tag{4.5}$$

Equation (4.5) predicts that the overall moduli reduce to zero at $p = 1/2$ and indicate the need for some caution in applying the effective medium method at high concentrations of inclusions with such extreme properties. The same can be said about the absolutely rigid spheres in an incompressible matrix, for which

$$\mu^* = \frac{\mu_0}{1 - \frac{5}{2}p} \tag{4.6}$$

Hence, in this case the modulus of μ^* becomes infinite when $p = 2/5$.

Let us note in the conclusion that the expressions (4.3) can be transformed to the exact ones for dilute concentrations of inclusions p. These expressions that are linear with respect to p have the form

$$K^* = K_0 + p(K - K_0)\left[1 + \frac{3(K - K_0)}{3K_0 + 4\mu_0}\right]^{-1}$$

$$\mu^* = \mu_0 + p(\mu - \mu_0)\left[1 + \frac{6(\mu - \mu_0)(K_0 + 2\mu_0)}{5\mu_0(3K_0 + 4\mu_0)}\right]^{-1} \tag{4.7}$$

5 The effective field method

Another self-consistent scheme for solving homogenization problems is a based on a different assumption. Let us consider an arbitrary i-th inclusion that occupies the region V_i in a fixed typical realization of a random set of inhomogeneities. We denote by $\varepsilon^*_{(i)}(x)$ the local internal field acting on that inclusion. This field is defined in V_i and is composed of the external field $\varepsilon^0(x)$ and disturbances of the field due to surrounding inclusions (Fig. 8). The self-consistent scheme in which the interaction between the inclusions is taken into account by introducing a local external field acting on each inclusion is called *the effective field method.*

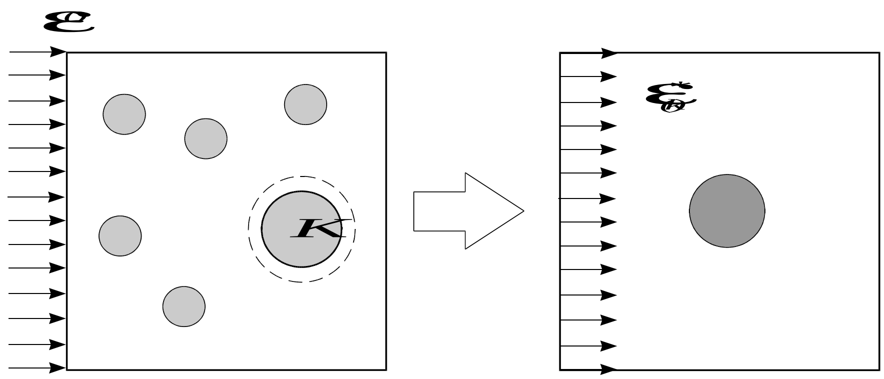

Fig. 8

Let now $\varepsilon^*(x)$ be the field which coincides with the $\varepsilon^*_{(i)}(x)$ when $x \in V_i$. With the help of the definition

$$V(x;x') = \sum_{k \neq i} V_k(x), \quad x \in V_i \tag{5.1}$$

we may write for an arbitrary point x inside the domain V

$$\varepsilon^*(x) = \varepsilon^0(x) + \int P(x-x')C^1(x')\varepsilon(x')V(x;x')dx' \tag{5.2}$$

where $C^1(x)$ coincides with the constant value C^1_i when $x \in V_i$. We suppose that the field $\varepsilon^*(x)$ has the same structure in any region occupied by inclusions (it is *the first main hypothesis of the effective field method*). In particular, if this field is constant inside each region V_i (but may vary randomly from one inclusion to another) the connection between the field $\varepsilon(x)$ ($x \in V$) and $\varepsilon^*(x)$ is given by the relation (2.66), which has been obtained above by solving the one-particle problem for an ellipsoidal inhomogeneity

$$\varepsilon(x) = A(x)\varepsilon^*(x) \tag{5.3}$$

Here $A(x)$ is the function which for $x \in V_i$ coincides with the constant tensor $A(a_i)$ which is defined by the expression

$$A(a_i) = \left[I + P(a_i)C_i^1\right]^{-1} \tag{5.4}$$

where a_i is as earlier the set of geometrical parameters that characterize the shape and orientation of the i-th ellipsoidal inclusion. In what follows we assume that every inclusion in the composite is in the local constant external field ε^* which depends on the geometrical characteristics a of this inclusion.

Substitution of the expression (5.3) into the right-hand side of equation (3.3) and (5.2) allows us to express the strain field at an arbitrary point of the medium via the local external field

$$\varepsilon(x) = \varepsilon^0(x) + \int P(x - x')C^A(x')\varepsilon^*(x')V(x')dx' \tag{5.5}$$

and to obtain the self-consistent equation for the field $\varepsilon^*(x)$

$$\varepsilon^*(x) = \varepsilon^0(x) + \int P(x - x')C^A(x')\varepsilon^*(x')V(x;x')dx' \tag{5.6}$$

where we have denoted

$$C^A(x) = C^1(x)A(x)$$

When we are concerned with a random set of inclusions, $\varepsilon(x)$ and $\varepsilon^*(x)$ are random functions. By taking the ensemble average of both sides of Eq. (5.5) we find

$$\langle\varepsilon(x)\rangle = \varepsilon^0(x) + \int P(x - x')\left\langle C^A(x')\varepsilon^*(x')V(x')|x'\right\rangle dx' \tag{5.7}$$

Here the symbol $\langle\cdot|x\rangle$ denotes the ensemble mean under the condition that the point x is located in the region V, occupied by the inclusions.

Let us now suppose that the random field $\varepsilon^*(x)$ in the points of the region V_i is statistically independent of the physical properties of this region (*the second main hypothesis of the effective field method*). This allows us to express the mean on the right-hand side of (5.7) as

$$\left\langle C^A(x')\varepsilon^*(x')V(x')|x'\right\rangle = \left\langle C^A(x)V(x)\widehat{\varepsilon}^*(x,a)\right\rangle \tag{5.8}$$

Here we have defined

$$\widehat{\varepsilon}^*(x,a) = \langle\varepsilon^*(x)|x,a\rangle \tag{5.9}$$

where the symbol $\langle\cdot|x,a\rangle$ denotes the mean under the condition that the point x is located in an inclusion with the characteristics a. The mean $\widehat{\varepsilon}^*(x,a)$ will be called the effective field acting on the inclusion with characteristics a.
Taking into account relation (5.8), Eq. (5.7) now takes the form

$$\langle\varepsilon(x)\rangle = \varepsilon^0(x) + \int P(x-x')T^*(x')dx' \tag{5.10}$$

where we have denoted

$$T^*(x) = \left\langle C^A(x)V(x)\widehat{\varepsilon}^*(x,a)\right\rangle \tag{5.11}$$

It follows from this that the average field $\langle\varepsilon(x)\rangle$ at an arbitrary point x of a composite material can be expressed by the moment of effective field $T^*(x')$. Eq. (5.6) gives the possibility to determine this moment. After averaging both parts of Eq. (5.6) under the condition $x \in V(a)$, we can write

$$\widehat{\varepsilon}^*(x,a) = \varepsilon^0(x) + \int P(x-x')\left\langle C^A(x')\varepsilon^*(x')V(x;x')|x';x,a\right\rangle dx' \tag{5.12}$$

Here the symbol $\langle\cdot|x';x,a\rangle$ denotes the operation of averaging under the condition $x \in V(a), \quad x' \in V$. In general the mean $\langle\cdot|x';x,a\rangle$ differs from $\langle\cdot|x,a\rangle$ and Eq. (5.12) turns out to be statistically unclosed. To obtain the former one must average Eq. (5.6) under a more complicated condition. Thus we obtain a hierarchy of equations connecting the conditional means of the effective field $\varepsilon^*(x)$. To close this hierarchy we must invoke certain additional assumptions concerning the statistical properties of the effective field. The simplest assumption is represented by the analog of the so-called "quasi-crystalline approximation", according to which the means $\langle\cdot|x';x,a\rangle$ and $\langle\cdot|x,a\rangle$ coincide. This results in

$$\widehat{\varepsilon}^*(x,a) = \varepsilon^0(x) + \int P(x-x')\left\langle C^A(x')\varepsilon^*(x')V(x;x')|x,a\right\rangle dx' \tag{5.13}$$

Assuming that the properties of the inclusions are statistically independent of their location in space, we may write the mean on the right-hand side of (5.13) in the form

$$\left\langle C^A(x')\varepsilon^*(x')V(x;x')|x,a\right\rangle = T^*(x')\Psi_a(x,x') \tag{5.14}$$

where

$$\Psi_a(x,x') = \frac{\langle V(x,x')|x,a\rangle}{\langle V(x)\rangle} \tag{5.15}$$

and definition of the conditional mean of a random function $f(x)$

$$\langle f(x)|x\rangle = \frac{\langle f(x)V(x)\rangle}{\langle V(x)\rangle}$$

has been taken into account.
For a spatially homogeneous set of inclusions the function $\Psi_a(x, x')$ depends only on the difference of arguments

$$\Psi_a(x, x') = \Psi_a(x - x') \tag{5.16}$$

It follows from the definition of the function $V(x, x')$ that $\Psi_a(x)$ is a continuous function and

$$\Psi_a(x) = 0, \quad \text{when} \quad x = 0 \tag{5.17}$$

Because of the weakening in geometrical linkage between the position of the inclusions when the distances between them increase, the following relation takes place

$$\Psi_a(x) \to 1, \quad \text{when} \quad |x| \to \infty \tag{5.18}$$

The function $\Psi_a(x)$ characterizes the density of inhomogeneity distribution surrounding the typical inclusion with characteristics a. It defines the shape of the "correlation hole", inside of which a typical inclusion "a" is located (Fig. 9).

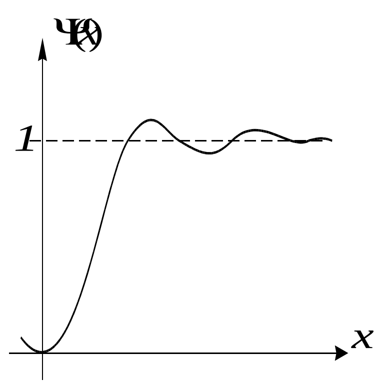

Fig. 9

If the set of inclusions possesses some symmetry (in the statistical sense) it influences the symmetry of the function $\Psi_a(x)$. Let us assume that there exists a linear transformation of x-space that sends the function $\Psi_a(x)$ into a spatially symmetric one

$$y = \alpha(a)x, \quad \Psi_a(\alpha^{-1}(a)y) = \Psi_a(|y|) \tag{5.19}$$

In this case the ellipsoid defined by the equation

$$|\alpha(a)x| = 1 \tag{5.20}$$

with semi-axes $\alpha_1, \alpha_2, \alpha_3$ describes the shape of the correlation hole.
Eq (5.13) takes the form

$$\widehat{\varepsilon}^*(x,a) = \varepsilon^0(x) + \int P(x-x')T^*(x')\Psi_a(x-x')dx' \tag{5.21}$$

Eliminating the external field $\varepsilon^0(x)$ from Eq. (5.10) and (5.21) we get the equation which couples the effective field $\widehat{\varepsilon}^*(x,a)$ and the average field $\langle\varepsilon(x)\rangle$ in the composite

$$\widehat{\varepsilon}^*(x,a) = \langle\varepsilon(x)\rangle - \int P(x-x')T^*(x')\Phi_a(x-x')dx' \tag{5.22}$$

$$\Phi_a(x) = 1 - \Psi_a(x)$$

For a spatially homogeneous random set of inclusions, $\Phi_a(x)$ is a smooth function, which quickly goes to zero outside a region having the order of the correlation hole size. If we neglect the change of the field $\widehat{\varepsilon}^*(x,a)$ in this region, Eq. (5.22) takes the form

$$\widehat{\varepsilon}^*(x,a) = \langle\varepsilon(x)\rangle + P_a^\Phi T^*(x) \tag{5.23}$$

$$P_a^\Phi = -\int P(x)\Phi_a(x)dx$$

Let us multiply both parts of Eq. (5.23) by the tensor $C^A(x)V(x)$ and average the result over the ensemble of random size and orientation of inclusions. This can be written as

$$T^*(x) = \overline{C}^A\langle\varepsilon(x)\rangle + \left\langle C^A(x)V(x)P_a^\Phi(x)\right\rangle T^*(x) \tag{5.24}$$

$$\overline{C}^A = \left\langle C^A(x)V(x)\right\rangle$$

where $P_a^\Phi(x)$ is the function which coincides with the constant tensor P_a^Φ calculated for the correlation hole of the inclusion with geometrical characteristics a.

Note that $C^A(x)$ is a homogeneous random function exhibiting the ergodic property. Using this property we obtain

$$\overline{C}^A = n_0 \left\langle vC^A(a)\right\rangle \tag{5.25}$$

Here n_0 is the numerical concentration of inclusions, v is the volume of the typical inclusion, and the averaging of the right part of this expression goes over the random sizes, orientation and properties of the ellipsoidal inhomogeneities.

Solving Eq. (5.24) for $T^*(x)$ we find

$$T^*(x) = D\overline{C}^A \langle\varepsilon(x)\rangle\,, \quad D = \left(I - \left\langle C^A(x)V(x)P_a^\Phi(x)\right\rangle\right)^{-1} \tag{5.26}$$

Substitution of this expression into the right-hand side of (5.10) gives

$$\langle\varepsilon(x)\rangle = \varepsilon^0(x) + \int P(x-x')D\overline{C}^A \langle\varepsilon(x')\rangle \, dx' \tag{5.27}$$

Let us apply the operator $\partial_j C^0_{ijkl}$ to both sides of this expression. Taking into account the relations

$$\partial_j C^0_{ijkl}\partial_l u^0_k(x) = 0, \quad \partial_j C^0_{ijkl}\partial_l G_{km}(x) = -\delta(x)\delta_{im} \tag{5.28}$$

we find that the average elastic fields $\langle u_i\rangle$ in the composite material satisfy the equation

$$\nabla C^* \nabla \langle u\rangle = 0, \quad C^* = C^0 + D\overline{C}^A \tag{5.29}$$

which coincides in form with the equilibrium equation of the elasticity theory for some homogeneous medium. The response of this medium to an external action is macroscopically identical with the reaction of a micro-inhomogeneous material. The tensor C^* represents the tensor of effective elastic moduli of the composite material.

In the frame of the considered scheme it is possible to evaluate not only the effective elastic moduli but also more detailed characteristics of the elastic fields in the composites. In accordance with the main hypothesis of the effective field method each inclusion behaves as isolated in the matrix. >From the solution of the one-particle problem we can find the expression for the strain field inside of the inclusion: $\varepsilon^+ = A\varepsilon^*$. Replacing the local external field ε^* in this expression by the effective field $\widehat{\varepsilon}^*(x,a)$ we have

$$\varepsilon^+ = A\widehat{\varepsilon}^*(x,a) \tag{5.30}$$

The Eqs. (5.23) and (5.26) allow us to write

$$\widehat{\varepsilon}^*(x,a) = (I + D\overline{C}^A)\,\langle\varepsilon(x)\rangle \tag{5.31}$$

Using now the expression (5.3) we can obtain the strain concentration on the inclusion in the composite material in the form

$$\varepsilon^- = F^\varepsilon(n_0)(I + D\overline{C}^A)\,\langle\varepsilon(x)\rangle \tag{5.32}$$

We consider now some special cases.

6 Some particular cases

6.1 The random set of spherical inclusions

Let all inclusions have the same elastic properties and be spherical of one and the same radius and both materials of the matrix and inclusions be isotropic with bulk moduli K_0, K and shear moduli μ_0, μ respectively. We assume that the shape of the correlation hole is also spherical. In this case we have

$$P_a^\Phi = P = (P_{ijkl}) = \frac{1}{9K_P}\delta_{ij}\delta_{kl} + \frac{1}{2\mu_P}\left(I_{ijkl} - \frac{1}{3}\delta_{ij}\delta_{kl}\right) \tag{6.1}$$

$$K_P = K_0 + \frac{4}{3}\mu_0, \quad \mu_P = \frac{5\mu_0(3K_0 + 4\mu_0)}{6(K_0 + 2\mu_0)}$$

The expression for the tensor of effective elastic moduli C^* can be written in the form

$$C^* = C^0 + p\left[(C^1)^{-1} + (1-p)P\right]^{-1} \tag{6.2}$$

or in more detail

$$C^*_{ijkl} = K^*\delta_{ij}\delta_{kl} + 2\mu^*\left(I_{ijkl} - \frac{1}{3}\delta_{ij}\delta_{kl}\right) \tag{6.3}$$

$$K^* = K_0 + p\left(\frac{1}{K_1} + \frac{1-p}{K_P}\right)^{-1}, \quad \mu^* = \mu_0 + p\left(\frac{1}{\mu_1} + \frac{1-p}{\mu_P}\right)^{-1}$$

where $K_1 = K - K_0, \quad \mu_1 = \mu - \mu_0$.
The formal transition to the limit $p = 1$ in these formulae leads to a physically reasonable result: $K^* \to K, \quad \mu^* \to \mu$. Thus these formulae are non-contradictory for any concentration of inclusions.
In particular, if the inclusions are holes ($K = 0, \mu = 0$), we have

$$K^* = K_0 - p\left(\frac{1}{K_0} - \frac{1-p}{K_P}\right)^{-1}, \quad \mu^* = \mu_0 - p\left(\frac{1}{\mu_0} - \frac{1-p}{\mu_P}\right)^{-1} \tag{6.4}$$

In the opposite case, if the inclusions are absolutely rigid spheres, then

$$K^* = K_0 + \frac{p}{1-p}K_P, \quad \mu^* = \mu_0 + \frac{p}{1-p}\mu_P \tag{6.5}$$

Let us note that in the case of an incompressible matrix ($K_0 = \infty$) Eq. (6.5) for μ^* gives

$$\mu^* = \mu_0\left[1 + \frac{5p}{2(1-p)}\right] \tag{6.6}$$

This expression coincides with the coefficient of viscosity of a suspension of rigid spheres in a viscous incompressible fluid. In the case of dilute suspension Eq. (6.6) is transformed to

$$\mu^* = \mu_0\left(1 + \frac{5}{2}p\right) \tag{6.7}$$

It is the famous result obtained by Einstein in the beginning of the century, it is the first result in the field of the mechanics of composite materials.
We have to note in the conclusion that the shape of the correlation hole may differ from the shape of spherical inclusions describing the so-called texture of composite. In this case the overall properties of composite material are not isotropic and their symmetry is determined by the ellipsoidal shape of the correlation hole.

6.2 The composite material reinforced by spheroidal inclusions

We consider now a composite material that consists of a homogeneous isotropic matrix in which spheroidal isotropic inclusions of an other component are distributed. The only geometrical parameters characterizing such inclusions are the aspect ratio $\gamma = a/a_3$ (a, a_3 are the spheroid semi-axes) and their orientation in space that can be determined by the unit vector m of the a_3-direction. It is convenient to write the tensor C^A in T-basis introduced above

$$C^A = k_A T^2 + 2m_A \left(T^1 - \frac{1}{2}T^2\right) + l_A(T^3 + T^4) + 4\mu_A T^5 + n_A T^6 \qquad (6.8)$$

$$k_A = \frac{1}{2\Delta}\left[\frac{\lambda_1 + \mu_1}{\mu_1(3\lambda_1 + 2\mu_1)} + P_6\right], \quad l_A = \frac{1}{\Delta}\left[\frac{\lambda_1}{2\mu_1(3\lambda_1 + 2\mu_1)} + P_3\right]$$

$$m_A = \frac{1}{2}\left(\frac{1}{2\mu_1} + P_2\right)^{-1}, \quad \mu_A = \left(\frac{1}{\mu_1} + P_5\right)^{-1}, \quad n_A = \frac{1}{\Delta}\left[\frac{\lambda_1 + 2\mu_1}{2\mu_1(3\lambda_1 + 2\mu_1)} + 2P_1\right]$$

$$\Delta = \frac{1}{2\mu_1(3\lambda_1 + 2\mu_1)}\left[1 + (\lambda_1 + 2\mu_1)P_6 + 4(\lambda_1 + \mu_1)P_1 + 4\lambda_1 P_3\right] + 2P_1P_6 - 2P_3^2$$

where the values $P_1, ..., P_6$ are determined in expressions (2.71) and depend on the aspect ratio γ , whereas the components of the tensor basis $T^1(m), ..., T^6(m)$ depend on the spheroid orientation m.
We assume now that the function $\Phi_a^P(x)$ has the symmetry a spheroid which is coaxial with the considered inclusion. Let the semi-axes of this spheroid be $\alpha_1 = \alpha_2 = \alpha$ and α_3 (the axis x_3 is directed along the vector m). In this case the tensor $P_a^\Phi(m)$ from (5.23) has the same form (2.71)

$$P_a^\Phi = P_1^\Phi T^2(m) + P_2^\Phi\left(T^1(m) - \frac{1}{2}T^2(m)\right) + P_3^\Phi(T^3(m) + T^4(m)) +$$

$$+P_5^\Phi T^5(m) + P_6^\Phi T^6(m) \qquad (6.9)$$

$$P_1^\Phi = \frac{1}{2\mu_0}\left[(1 - \kappa_0)f_0 + f_1\right], \quad P_2^\Phi = \frac{1}{2\mu_0}\left[(2 - \kappa_0)f_0 + f_1\right], \quad P_3^\Phi = -\frac{1}{\mu_0}f_1$$

$$P_2^\Phi = \frac{1}{\mu_0}(1 - f_0 - 4f_1), \quad P_2^\Phi = \frac{1}{\mu_0}\left[(1 - \kappa_0)(1 - 2f_0) + 2f_1\right], \quad f_0 = \frac{1 - g}{2(1 - \zeta^2)}$$

$$f_1 = \frac{\kappa_0}{4(1-\zeta^2)^2}[(2+\zeta^2)g - 3\zeta^2], \quad g = \frac{\zeta^2}{\sqrt{\zeta^2-1}} \arctan\sqrt{\zeta^2-1}, \quad \zeta = \frac{\alpha}{\alpha_3} > 1$$

with generally other aspect ratio $\zeta = \alpha/\alpha_3$. If the spatial positions of the inclusions are statistically independent (that will be assumed below) the parameter ζ has the order $\langle a/a_3 \rangle$. We assume for simplicity that all inclusions have the same sizes, aspect ratio and properties. In this case the tensors $P_a^{\Phi}(m)$ and $P(m)$ from (2.71) coincide and the general expression (5.29) can be rewritten in the form

$$C^* = C^0 + n_0 v \left(I - n_0 v \left\langle C^A(m)P(m)\right\rangle\right)^{-1} \left\langle C^A(m)\right\rangle. \tag{6.10}$$

Here the means $\left\langle C^A(m)P(m)\right\rangle$ and $\left\langle C^A(m)\right\rangle$ are calculated over the ensemble of inclusion distribution by orientation, n_0 is the numerical concentration of inclusions, $v = \frac{4}{3}\pi a^2 a_3$ is the inclusion volume.

6.3 Elastic medium reinforced by stiff flakes

In this section we apply the above results to the calculation of elastic moduli of composites reinforced by stiff ($k, \mu \to \infty$) inclusions having the shape of flattened spheroids. In this case the tensor $C^A(m)$ is determined by the expression

$$C^A(m) = C^1(I + P(m)C^1)^{-1} = P^{-1}(m). \tag{6.11}$$

For the strongly oblate spheroid ($\gamma \to \infty$) we have to use the asymptotic expansion of the function $f_0(\gamma)$ and $f_1(\gamma)$ from (2.72). Taking into account only the principal terms of this expansion

$$f_0(\gamma) = \frac{\pi}{4\gamma}, \quad f_1(\gamma) = \frac{\kappa_0 \pi}{8\gamma}, \quad \kappa_0 = \frac{\lambda_0 + \mu_0}{\lambda_0 + 2\mu_0} \tag{6.12}$$

we have

$$P_1 = \frac{\pi(2-\kappa_0)}{16\mu_0\gamma}, \quad P_2 = \frac{\pi(4-\kappa_0)}{16\mu_0\gamma}, \quad P_3 = -\frac{\pi\kappa_0}{8\mu_0\gamma} \tag{6.13}$$

$$P_5 = \frac{1}{\mu_0}\left[1 - \frac{\pi}{4\gamma}(1+2\kappa_0)\right], \quad P_6 = \frac{1}{\mu_0}\left[1 - \kappa_0 - \frac{\pi}{4\gamma}(2-3\kappa_0)\right].$$

Calculating the inverse expression for the tensor P with the help of Eq. (2.74) we obtain

$$C^A(m) = \left[k_A T^2(m) + 2m_A\left(T^1(m) - \frac{1}{2}T^2(m)\right)\right]\gamma + O(1) \tag{6.14}$$

$$k_A = \frac{4\mu_0}{\pi(2-\kappa_0)}, \quad m_A = \frac{8\mu_0}{\pi(4-\kappa_0)}.$$

Let the inclusions be thin spheroids (flakes) of one and the same orientation. According to Eq. (6.10) in this case the composite will be transversely isotropic with the axis of isotropy directed along the vector m. Eq. (6.10) for the tensor of effective moduli of elasticity C^* takes the form

$$C^* = C^0 + \frac{\langle\tau\rangle k_A}{1-\langle\tau\rangle k_A P_1^\Phi} T^2(m) + \frac{2\langle\tau\rangle m_A}{1-\langle\tau\rangle m_A P_2^\Phi}\left[T^1(m) - \frac{1}{2}T^2(m)\right] \quad (6.15)$$

where we have denoted

$$\tau = \frac{4}{3} n_0 \pi a^3. \quad (6.16)$$

If $\zeta \gg 1$, then this expression transforms to

$$C^* = C^0 + \langle\tau\rangle k_A \left(1 - \frac{\langle\tau\rangle}{\zeta}\right)^{-1} T^2(m) +$$

$$+2\langle\tau\rangle m_A \left(1 - \frac{\langle\tau\rangle}{\zeta}\right)^{-1}\left[T^1(m) - \frac{1}{2}T^2(m)\right]. \quad (6.17)$$

Let us consider now rigid discs homogeneously distributed over the orientation. In this case we have to perform the average operation in Eq. (6.10). In order to simplify this operation we present here first of all the result of T-basis tensors averaging over the orientation

$$\langle T^1(m)\rangle = \frac{1}{45}\left(10E^1 + 21E^2\right), \quad \langle T^1(m)\rangle = \frac{2}{45}\left(10E^1 + 3E^2\right)$$

$$\langle T^3(m)\rangle = \langle T^4(m)\rangle = \frac{2}{45}\left(5E^1 - 3E^2\right), \quad \langle T^3(m)\rangle = \frac{1}{5}E^2 \quad (6.18)$$

$$\langle T^6(m)\rangle = \frac{1}{45}\left(5E^1 + 6E^2\right)$$

where we have denoted

$$E^1_{ijkl} = \delta_{ij}\delta_{kl}, \quad E^2_{ijkl} = I_{ijkl} - \frac{1}{3}\delta_{ij}\delta_{kl}.$$

The composite material reinforced by rigid flakes with disordered orientation in space is macroscopically isotropic and with the help of Eq. (6.15) we obtain the following expression for the tensor of effective elastic moduli C^*

$$C^* = K^* E^1 + 2\mu^* E^2 \quad (6.19)$$

$$K^* = K_0 + \frac{4}{3} \langle\tau\rangle k_A \left[3 - 4 \langle\tau\rangle k_A (2P_1^\Phi + P_3^\Phi)\right]^{-1}$$

$$\mu^* = \mu_0 + \langle\tau\rangle (k_A + 6m_A) \left\{15 - 4 \langle\tau\rangle \left[k_A(P_1^\Phi - P_3^\Phi) + 3m_A P_2^\Phi\right]\right\}^{-1}.$$

Note that the case of non-interacting inclusions follows from Eqs (6.15) and (6.19) when $\gamma \to \infty$ and $P_1^\Phi = P_2^\Phi = P_3^\Phi = 0$.

6.4 Penny-shaped cracks in an elastic medium

Next we examine the case of thin spheroids with elastic moduli essentially smaller than the moduli of the medium. To solve the homogenization problem for the medium with such inclusions it is convenient to fix the external stress field ($\langle\sigma\rangle = \sigma^0$). In this case the actions of the integral operators with the kernels $P(x)$ and $Q(x)$ are determined by expressions (3.6).

Equations (3.11) and (3.12) can be rewritten in the form

$$\varepsilon(x) = \varepsilon^0(x) + \int P(x - x')C^0 S^1(x')\sigma(x')dx' \tag{6.20}$$

$$\sigma(x) = \sigma^0(x) + \int Q(x - x')S^1(x')\sigma(x')dx'. \tag{6.21}$$

As before we assume that every inclusion with geometrical characteristics a is located in a constant local external field $\sigma^*(x)$. Hence, we can write

$$\sigma(x) = B(x)\sigma^*(x) \tag{6.22}$$

where $B(x)$ is the function that coincides with the constant tensor

$$B(a_i) = \left[I + Q(a_i)S^1\right]^{-1} \tag{6.23}$$

when $x \in V_i$.

Substitution of relation (6.22) into the right-hand sides of Eqs (6.20) and (6.21) gives

$$\varepsilon(x) = \varepsilon^0(x) + \int P(x - x')C^0 S^B(x')\sigma^*(x')V(x')dx' \tag{6.24}$$

$$\sigma(x) = \sigma^0(x) + \int Q(x - x')S^B(x')\sigma^*(x')V(x')dx' \tag{6.25}$$

$$S^B(x) = S^1(x)B(x)$$

Let us apply these equations to the spheroidal inclusions with elastic moduli that are equal to zero (spheroidal pores). In this case we have

$$S^B(x) = Q^{-1}(x), \quad Q(x) = Q(a_i), \quad \text{when} \quad x \in V_i \tag{6.26}$$

where the constant tensor $Q(a_i)$ is determined in Eq. (2.75). If $\gamma \gg 1$ such inclusions can model penny-shaped cracks in the elastic medium. For the crack the tensor Q takes the form (2.76). Calculating the inverse tensor with the help of Eq. (2.74) we obtain that Q^{-1} depends on the orientation of vector m and is determined by the expression

$$Q^{-1}(m) = M(m) = \frac{\gamma}{\pi}\left[M_5 T^5(m) + M_6 T^6(m)\right] + O(1) \tag{6.27}$$

$$M_5 = \frac{4}{\mu_0(1+2\kappa_0)}, \quad M_6 = \frac{1}{\mu_0(1+\kappa_0)}$$

We consider now the simplest variant of the effective field method in which we assume that every inclusion in the composite is in a local homogeneous external field σ^* that depends only on the orientation m of the inclusion (Fig. 10).

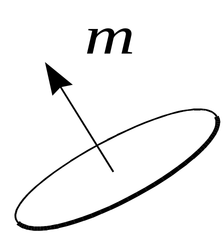

Fig. 10

The equation for the effective field $\widehat{\sigma}^*(m)$ can be obtained in the same way as Eq. (5.13) and has the form

$$\widehat{\sigma}^*(m) = \sigma^0 + \int Q(x-x')\left\langle M(m')\sigma^*(m')V(x;x')|x,m\right\rangle dx' \tag{6.28}$$

where we have denoted: $m' = m(x')$. The mean value of the integrand in this equation is

$$\left\langle M(m')\sigma^*(m')V(x;x')|x,m\right\rangle = \left\langle \overline{M(m)}\widehat{\sigma}^*(m)\right\rangle \Psi_m(x-x') \tag{6.29}$$

where $\Psi_m(x)$ is determined above and we have denoted

$$\overline{M}(m) = \langle\tau\rangle \left[M_5 T^5(m) + M_6 T^6(m)\right], \quad \tau = \frac{4}{3} n_0 a^3. \tag{6.30}$$

In what follows the function $\Psi_m(x)$ is supposed to have the symmetry of a spheroid coaxial with the inclusion of orientation m. Taking into account the properties of $\Psi_m(x)$ one can calculate the integral in the right-hand side of Eq. (6.28) and obtain the expression for $\widehat{\sigma}^*(m)$ in the form

$$\widehat{\sigma}^*(m) = \sigma^0 + Q_m^\Phi \left\langle \overline{M}(m)\widehat{\sigma}^*(m)\right\rangle \tag{6.31}$$

where

$$Q_m^\Phi = \int Q(x)(1 - \Psi_m(x))dx, \quad Q_m^\Phi = C^0 - C^0 P_m^\Phi C^0 \tag{6.32}$$

where P_m^Φ is determined by the Eq. (6.9).

Let us multiply both parts of Eq. (6.31) by the tensor $M(m)$ and average the result over the ensemble distributions of orientations and sizes of inclusions. In such a manner the equation for the mean value $\left\langle \overline{M}(m)\widehat{\sigma}^*(m)\right\rangle$ can be obtained and its solution has the form

$$\left\langle \overline{M}(m)\widehat{\sigma}^*(m)\right\rangle = \left[I - \left\langle \overline{M}(m)Q_m^\Phi\right\rangle\right]^{-1} \left\langle \overline{M}(m)\right\rangle \sigma^0 \tag{6.33}$$

After averaging Eqs. (6.24) and (6.25) for the strains and stresses and taking into account the relation

$$\langle M(x)\sigma^*(x)V(x)\rangle = \left\langle \overline{M}(m)\widehat{\sigma}^*(m)\right\rangle \tag{6.34}$$

we obtain the result

$$\langle\varepsilon\rangle = \varepsilon^0 + \left[I - \left\langle \overline{M}(m)Q_m^\Phi\right\rangle\right]^{-1} \left\langle \overline{M}(m)\right\rangle \sigma^0, \quad \sigma^0 = \langle\sigma\rangle$$

Since $\varepsilon^0 = S^0\sigma^0$ we can recast these relations in the form

$$\langle\varepsilon\rangle = S^* \langle\sigma\rangle, \quad S^* = S^0 + \left[I - \left\langle \overline{M}(m)Q_m^\Phi\right\rangle\right]^{-1} \left\langle \overline{M}(m)\right\rangle \tag{6.35}$$

where S^* is the tensor of elastic compliances of the medium with penny-shaped cracks.

Let the medium be isotropic and the function $\Psi_m(m)$ have the symmetry of the spheroid with semi-axes $\alpha_1 = \alpha_2 = \alpha, \quad \alpha_3$. This spheroid is coaxial with the inclusion of the orientation m. In this case the representation of the tensor Q_m^Φ in the T-basis has the form

$$Q_m^\Phi = q_1 T^2(m) + q_2\left(T^1(m) - \frac{1}{2}T^2(m)\right) + q_3(T^3(m) + T^4(m)) +$$

$$+q_5 T^5(m) + q_6 T^6(m) \tag{6.36}$$

$$q_1 = \mu_0[1 - 4\kappa_0 - 2(1 - 3\kappa_0)f_0 + 2\kappa_0 f_1], \quad q_2 = 2\mu_0[1 + (2 - \kappa_0)f_0 + \kappa_0 f_1]$$

$$q_3 = 2\mu_0[(1 - 2\kappa_0)f_0 - 2\kappa_0 f_1], \quad q_5 = -4\mu_0(f_0 + 4\kappa_0 f_1)$$

$$q_6 = -4\mu_0[(1 + 2\kappa_0)f_0 - 2\kappa_0 f_1]$$

where the functions $f_0(\zeta)$ and $f_1(\zeta)$ ($\zeta = \alpha/\alpha_3 > 1$) are the same as in Eq. (6.9). If $\zeta \gg 1$ the coefficients $q_i, \quad i = 1, 2, ..., 6$ in Eq. (6.36) with accuracy of order ζ^{-1} are transformed into

$$q_1 = \mu_0(1 - 4\kappa_0) + \frac{\pi\mu_0}{4\zeta}(7\kappa_0 - 2), \quad q_2 = -2\mu_0 + \frac{\pi\mu_0}{4\zeta}(4 - \kappa_0)$$

$$q_3 = -\frac{\pi\mu_0}{2\zeta}(3\kappa_0 - 1), \quad q_5 = -\frac{\pi\mu_0}{\zeta}(1 + 2\kappa_0), \quad q_6 = -\frac{\pi\mu_0}{\zeta}(1 + \kappa_0). \tag{6.37}$$

In the limit $\zeta \to \infty$ we have

$$Q_m^\Phi = -2\mu_0[T^1(m) - (1 - 2\kappa_0)T^2(m)] \tag{6.38}$$

The other limit ($\zeta \to 1$) corresponds to a correlation hole of the shape of a sphere. In that case Q_m^Φ is the isotropic tensor

$$Q_m^\Phi = \frac{4}{9}\mu_0[(1 - 4\kappa_0)E^1 - \frac{3}{9}(5 - 4\kappa_0)E^2] \tag{6.39}$$

where tensors E^1 and E^2 are determined earlier.
The case $\zeta = 1$ corresponds to the model of a random set of inclusions when there is a spherical area around every inclusion and the probability of other inclusions appearing in this area is small.
Let us consider the expressions for S^* (6.35) in some particular cases.

6.4.1 Penny-shaped cracks of the same orientation

It follows from Eqs. (6.35) and (6.36) that the tensor S^* has the form

$$S^* = S^0 + \frac{4\langle\tau\rangle M_5}{4 - \langle\tau\rangle M_5 q_5} T^5(m) + \frac{\langle\tau\rangle M_6}{1 - \langle\tau\rangle M_6 q_6} T^6(m) \tag{6.40}$$

In the case of $\zeta \gg 1$ the expression for S^* becomes

$$S^* = S^0 + \left(1 - \frac{\langle\tau\rangle}{\zeta}\right)^{-1} \overline{M}(m) \tag{6.41}$$

The limit $\zeta \to \infty$ here gives the formulae which correspond to the case of non-interacting inclusions.

6.4.2 Homogeneous distribution of cracks over the orientation

The tensor S^* in this case is isotropic and has the form

$$S^* = S^0 + \frac{\langle\tau\rangle M_6}{9(1 - \langle\tau\rangle j_1)} E^1 + \frac{\langle\tau\rangle (3M_5 + 2M_6)}{15(1 - \langle\tau\rangle j_2)} E^2 \tag{6.42}$$

$$j_1 = \frac{1}{3} M_6 (2q_3 + q_6), \quad j_2 = \frac{1}{10} M_5 q_5 + \frac{2}{15} M_6 (q_6 - q_3)$$

The expressions for the bulk K^* and shear μ^* moduli for the cracked solid follow from this result:

$$K^* = K_0 \left[1 + \frac{\langle\tau\rangle K_0 M_6}{1 - \langle\tau\rangle j_1}\right]^{-1} \tag{6.43}$$

$$\mu^* = \mu_0 \left[1 + \frac{2\mu_0}{15(1 - \langle\tau\rangle j_2)} (3M_5 + 2M_6)\right]^{-1}.$$

If $\zeta = 1$ these expressions take the form

$$K^* = K_0[1 - M_K(1 + s_1 M_K)^{-1}], \quad \mu^* = \mu_0[1 - M_\mu(1 + s_2 M_\mu)^{-1}] \tag{6.44}$$

where:

$$M_K = \langle\tau\rangle K_0 M_6, \quad M_\mu = \frac{2}{15} \langle\tau\rangle \mu_0 (3M_5 + 2M_6)$$

$$s_1 = \frac{3K_0}{3K_0 + 4\mu_0}, \quad s_2 = \frac{1}{5}(3 - s_1)$$

6.5 The fiber reinforced composite material

We consider now the composite material whose matrix is transversely isotropic. The tensor S^0 for such a medium can be represented in the form (2.81). Let us suppose the inclusions in the composite have the form of continuous cylinders of the same radius similarly oriented parallel to the axis of symmetry of the elastic properties of the matrix (a medium reinforced by unidirectional continuous fibers).

As it was mentioned above, the solution of the one-particle problem in this case can be obtained from the solution for the prolate spheroid by the limiting transition when the diameter of the spheroid is fixed and other semi-axes tends to infinity. The tensor C^A if the fibers are also transversely isotropic is determined as

$$C^A = k_1 \left(1 + \frac{k_1}{k_0 + m_0}\right)^{-1} T^2 + 2m_1 \left[1 + \frac{m_1(k_0 + 2m_0)}{2m_0(k_0 + m_0)}\right]^{-1} \left(T^1 - \frac{1}{2}T^2\right) +$$

$$+ l_1 \left(1 + \frac{k_1}{k_0 + m_0}\right)^{-1} (T^3 + T^4) + 2\mu_1 \left(1 + \frac{\mu_1}{2\mu_0}\right)^{-1} T^5 + \tag{6.45}$$

$$+ \left[n_1 - \frac{l_1^2}{k_0 + m_0} \left(1 + \frac{k_1}{k_0 + m_0}\right)^{-1}\right] T^6$$

If we assume that the shape of the correlation hole is cylindrical coaxial with the fiber (P^Φ in this case coincides with tensor P from Eq. (2.82)), the effective elastic moduli tensor C^* takes the form

$$C^* = C^0 + pC^1 \left[I + (1 - p)C^1 P\right]^{-1} \tag{6.46}$$

where p is the fiber volume concentration. The expression (6.46) gives in detail

$$C^* = k^* T^2 + 2m^* \left(T^1 + \frac{1}{2}T^2\right) + l^*(T^3 + T^4) + 4\mu^* T^5 + n^* T^6 \tag{6.47}$$

$$k^* = k_0 + pk_1 \left[1 + \frac{(1-p)k_1}{k_0 + m_0}\right]^{-1}, \quad m^* = m_0 + pm_1 \left[1 + (1-p)\frac{m_1(k_0 + 2m_0)}{2m_0(k_0 + m_0)}\right]^{-1}$$

$$l^* = l_0 + pl_1 \left[1 + \frac{(1-p)k_1}{k_0 + m_0} \right]^{-1}, \quad \mu^* = \mu_0 + p\mu_1 \left[1 + (1-p)\frac{\mu_1}{2\mu_0} \right]^{-1}$$

$$n^* = n_0 + p \left[n_1 - \frac{(1-p)l_1^2}{k_0 + m_0} \left(1 + \frac{(1-p)k_1}{k_0 + m_0} \right)^{-1} \right].$$

The formal transition to the limit $p \to 1$ in these formulae leads to a physically reasonable result: all effective elastic moduli of the fiber reinforced material become the moduli of the fibers. Thus these formulae are non-contradictory in the entire region of fiber volume concentrations.

7 Thermal deformation of matrix composites

Let us consider thermoelastic deformation of composite materials containing a random set of ellipsoidal inclusions. We will assume that the temperature field T is constant ($T = 1$) and deformation of the composite is not constrained at infinity. According to the effective field method each inclusion in the composite is supposed to behave as an isolated one in a homogeneous matrix with tensor of elastic moduli C^0 and thermal expansion coefficients α^0 . The presence of surrounding inclusions is taken into account through the effective external field $\sigma^*(x)$ affecting every inclusion.
Thus the thermo-stresses $\sigma(x)$ in a region V_i occupied by the i-th inclusion satisfy the following equation

$$\sigma(x) - \int_{V_i} Q(x - x')S_i^1 \sigma(x')dx' = \sigma^*(x) + Q(a_i)\alpha_i^1 \tag{7.1}$$

$$S_i^1 = S^i - S^0, \quad \alpha_i^1 = \alpha^i - \alpha^0$$

where S_i^1 and α_i^1 are the disturbances of the elastic compliances and thermal expansion coefficients in V_i ; the tensor $Q(a_i)$ is defined in (2.67).
The solution of this equation can be represented as follows:

$$\sigma(x) = \sigma^s(x) + \sigma^T(x) \tag{7.2}$$

where the functions $\sigma^s(x)$ and $\sigma^T(x)$ satisfy Eq. (7.1) with the right-hand side $\sigma^*(x)$ and $Q(a_i)\alpha_i^1$ respectively. If the effective stress field σ^* is assumed to be constant then σ^s and σ^T for the ellipsoidal inclusion are also constant

$$\sigma^s(x) = B(a_i)\sigma^*, \quad \sigma^T = -B(a_i)Q(a_i)\alpha_i^1 \tag{7.3}$$

where the tensor $B(a_i)$ is defined in (2.67).
Using these results one can evaluate the stress and strains in the medium with a set of inclusions occupying the region V in the temperature field $T = 1$

$$\sigma(x) = \int Q(x - x') \left[S^1(x')\sigma(x') + \alpha^1(x')\right] V(x')dx' \tag{7.4}$$

$$\varepsilon(x) = \varepsilon^0 + \int P(x - x')C^0 \left[S^1(x')\sigma(x') + \alpha^1(x')\right] V(x')dx' \tag{7.5}$$

where the expression in square brackets takes the form

$$S^1(x)\sigma(x) + \alpha^1(x) = S^B(x)\sigma^*(x) + B^T(x) \tag{7.6}$$

$$S^B(x) = S^1(x)B(x), \quad B^T(x) = -S^B(x)Q(x)\alpha^1(x) + \alpha^1(x)$$

The equation for the mean value of the effective stress field σ^* follows from Eqs. (7.1) and (7.6), namely,

$$\sigma^* = \int Q(x - x') \left\langle [S^B(x')\sigma^* + B^T(x')]V(x; x')|x\right\rangle dx' \tag{7.7}$$

where the functions $S^B(x')$ and $B^T(x')$ are constant inside the inclusions.
Assuming statistical independence of spatial distribution and elastic properties of the inclusions, we can represent the mean value of the integrand in Eq. (7.7) as

$$\left\langle [S^B(x')\sigma^* + B^T(x')]V(x; x')|x\right\rangle = n_0(S^B\sigma^* + B^T)\Psi(x - x') \tag{7.8}$$

$$S^B = \left\langle \int_v S^B(x)dx \right\rangle, \quad B^T = \left\langle \int_v B^T(x)dx \right\rangle$$

where n_0 is the number density of the inclusions.
After substituting expressions (7.8) into (7.7) and calculating the integral we have

$$\sigma^* = -n_0 Q^\Phi(S^B\sigma^* + B^T), \quad Q^\Phi = \int Q(x)[1 - \Psi(x)]dx \tag{7.9}$$

where the relation $\langle\sigma(x)\rangle = 0$ was taken into account.
After solving Eq. (7.9) with respect to σ^* we get

$$\sigma^* = -n_0(I + n_0 Q^\Phi S^B)^{-1} Q^\Phi B^T \tag{7.10}$$

Let us average now Eq. (7.5) for the strain tensor ε

$$\langle \varepsilon \rangle = \alpha^0 + n_0(S^B \sigma^* + B^T) \tag{7.11}$$

This is the mean strain tensor of the composite material under the temperature change per degree. Thus $\langle \varepsilon \rangle$ coincides with the thermal expansion coefficients tensor α^*. In view of Eq. (7.10) we can write

$$\alpha^* = \alpha^0 + n_0(I + n_0 Q^\Phi S^B)^{-1} B^T \tag{7.12}$$

Let spherical inclusions in the composite have the same isotropic thermoelastic properties. If $\Psi(x)$ is a spherically symmetric function ($\Psi(x) = \Psi(|x|)$), then Eq. (7.12) takes the form

$$\alpha^*_{ij} = \alpha^* \delta_{ij}, \quad \alpha^* = \alpha_0 + p(\alpha - \alpha_0)\left[1 - (1-p)\frac{4\mu_0(k - k_0)}{k(3k_0 + 4\mu_0)}\right]^{-1} \tag{7.13}$$

There is another way to determine the tensor α^* which establishes a curious cross-property relation that is absent for the common homogeneous medium. We consider for simplicity a two-phase medium, that is under the action of two types of straining: the first, due to a given surface traction at fixed temperature, the second due to temperature change with no such tractions.
More specifically, let $\varepsilon(x)$ and $\sigma(x)$ be the strain and stress fields that appear in the volume V at fixed temperature $T = 0$ and under the homogeneous boundary conditions: $F_i = \sigma^0_{ij} n_j$, σ^0_{ij} =const. Because of

$$\frac{1}{V}\int_V \sigma_{ij} dV = \frac{1}{2V}\int_{\partial V} (F_i x_j + F_j x_i) ds = \sigma^0_{ij} \tag{7.14}$$

these condition imply

$$\langle \varepsilon \rangle = S^* \sigma^0 \tag{7.15}$$

where S^* is the effective compliance tensor of the medium.
Consider a second pair of strain and stress fields in the medium $\varepsilon^{(T)}(x)$ and $\sigma^{(T)}(x)$ due to the temperature change T with no boundary traction; let $u^{(T)}(x)$ be the appropriate displacement field. These fields appear as a consequence of the inhomogeneity of the thermal expansion coefficient $\alpha(x)$ which takes the different values α_0 and α in the matrix and inclusions respectively. According to the basic thermoelastic law, we have

$$\varepsilon^{(T)}(x) = S(x)\sigma^{(T)}(x) + \alpha(x)T \tag{7.16}$$

Moreover

$$\left\langle \varepsilon^{(T)} \right\rangle = \frac{1}{V} \int\limits_V \varepsilon^{(T)}(x)dx = \alpha^* T \tag{7.17}$$

which is just the definition of the effective thermal expansion coefficient α^* of the medium. If we assume for simplicity that the composite material is macroscopically isotropic, then $\alpha^*_{ij} = \alpha^* \delta_{ij}$.
Note first the identity

$$\frac{1}{V} \int\limits_V \sigma_{ij}(x)\varepsilon_{ij}^{(T)}(x)dx = \frac{1}{V} \int\limits_S \nabla_j(\sigma_{ij}(x)u_i^{(T)}(x))dx =$$

$$= \frac{1}{V} \int\limits_S n_j \sigma_{ij}(x) u_i^{(T)}(x)dS = \sigma^0_{ij} \left\langle \varepsilon^{(T)}_{ij} \right\rangle = \alpha^* T \sigma^0_{kk} \tag{7.18}$$

which follows from Gauss' theorem, the boundary conditions for the stresses, the self-equilibrium of the field $\sigma(x)$ and the definition (7.17) of the effective thermal expansion constant.
The second identity that we shell need reads

$$\int\limits_V \sigma^{(T)}_{ij}(x)\varepsilon_{ij}(x)dx = \int\limits_S n_j \sigma^{(T)}_{ij}(x)u_i(x)dS =$$

$$= \int\limits_V \sigma^{(T)}_{ij}(x) S_{ijkl}(x)\sigma_{kl}(x)dx = 0 \tag{7.19}$$

since the field $\sigma^{(T)}(x)$ is "temperature-induced", with no tractions involved.
Introduce now Hook's law (7.16) into the first integral of (7.18)

$$\frac{1}{V} \int\limits_V \sigma_{ij}(x)[S_{ijkl}(x)\sigma^{(T)}_{kl}(x) + \alpha(x) T \delta_{ij}]dx =$$

$$= \frac{1}{V} T \int\limits_V \alpha(x)\sigma_{kk}(x)dx = \alpha^* \sigma^0_{kk} \tag{7.20}$$

having used (7.19). Since $\alpha(x)$ is step-constant, Eq. (7.19) gives

$$p\alpha \left\langle \sigma_{kk} \right\rangle_{inc} + (1-p)\alpha_0 \left\langle \sigma_{kk} \right\rangle_m = \alpha^* \sigma^0_{kk} \tag{7.21}$$

where $\langle\cdot\rangle_{inc}$ and $\langle\cdot\rangle_m$ denotes the average operation over the region occupied by inclusions and matrix respectively.
It remains to combine now (7.21) with the obvious formulae

$$\sigma_{kk}^0 = p\,\langle\sigma_{kk}\rangle_{inc} + (1-p)\,\langle\sigma_{kk}\rangle_m$$

$$\varepsilon_{kk}^0 = p\,\langle\varepsilon_{kk}\rangle_{inc} + (1-p)\,\langle\varepsilon_{kk}\rangle_m \tag{7.22}$$

$$\sigma_{kk}^0 = 3K^*\varepsilon_{kk}^0, \quad \langle\sigma_{kk}\rangle_{inc} = 3K\,\langle\varepsilon_{kk}\rangle_{inc}, \quad \langle\sigma_{kk}\rangle_m = 3K_0\,\langle\varepsilon_{kk}\rangle_m$$

which allow to eliminate all the traces of the stress and strain tensors. The final result

$$\alpha^* = \left[\alpha_0\left(\frac{1}{K^*}-\frac{1}{K}\right) - \alpha\left(\frac{1}{K^*}-\frac{1}{K_0}\right)\right]\left(\frac{1}{K^0}-\frac{1}{K}\right)^{-1} \tag{7.23}$$

expresses the effective coefficient of the thermal expansions α^* via the effective bulk modulus K^* of the same material.

8 Mori-Tanaka approach

An approach which, for composites, turned out to be closely connected, if not equivalent, to the effective field approximation in the literature on composites is often associated with the names of Mori-Tanaka. The original derivation of these authors was expressed in terms of eigenstrains, equivalent inclusion, transformation energy consideration etc. and at first glance looked totally different in spirit from the rest of the self-consistent approaches in mechanics of heterogeneous media. The clarification of the nature of this approximation is in the assumption that *every inclusion behaves as an isolated one in the matrix of the composite and is under the action of a constant (one and the same for all inclusions) external strain (or stress) field. This field is assumed to coincide with the average strain (or stress) field in the matrix.*
Let us consider for simplicity a two-phase composite in which the inclusions are identical spheres randomly distributed in the matrix. In this case we can write the obvious relations

$$\langle\sigma\rangle = p\,\langle\sigma\rangle_{inc} + (1-p)\,\langle\sigma\rangle_m, \quad \langle\varepsilon\rangle = p\,\langle\varepsilon\rangle_{inc} + (1-p)\,\langle\varepsilon\rangle_m \tag{8.1}$$

With the help of the local Hook's law we obtain

$$\langle\sigma\rangle = pC\,\langle\varepsilon\rangle_{inc} + (1-p)C^0\,\langle\varepsilon\rangle_m = C^0\,\langle\varepsilon\rangle + pC^1\,\langle\varepsilon\rangle_{inc} \tag{8.2}$$

If the Mori-Tanaka main assumption is accepted we can write

$$\langle\varepsilon\rangle_{inc} = A\,\langle\varepsilon\rangle_m\,, \quad A = (I + PC^1)^{-1} \tag{8.3}$$

where the tensor A was determined above for the spherical inclusion. Substitution of this relation into the second part of the Eq. (8.1) gives

$$\langle\varepsilon\rangle = (pA + (1-p)I)\,\langle\varepsilon\rangle_m \tag{8.4}$$

from which we obtain

$$\langle\varepsilon\rangle_m = (I - pPC^1A)^{-1}\,\langle\varepsilon\rangle \tag{8.5}$$

Putting the expression

$$\langle\varepsilon\rangle_{inc} = A(I - pPC^1A)^{-1}\,\langle\varepsilon\rangle \tag{8.6}$$

into the right-hand side of (9.2) we obtain finally

$$\langle\sigma\rangle = C^*\,\langle\varepsilon\rangle\,, \quad C^* = C^0 + pC^A(I - pPC^A)^{-1} \tag{8.7}$$

with the notation $C^A = C^1A$. It is obvious that these formulae coincide with the analogous expressions obtained by the effective field method if we accept the additional assumption that the shape of the correlation hole coincides with the shape of a typical inclusion. In the general case these shapes can be different and the tensors P and

$$P^\Phi = \int P(x)\Phi(x)dx$$

are not the same.

This result can be generalized very easily to the case of multi-phase systems and non-spherical inclusions.

The attractive feature of this approach is its extreme simplicity. At the same time it has to be noted that Mori-Tanaka scheme works well and produces reasonable results in the cases when the inclusions are either spherical or, if non-spherical, are aligned. For multi-phase systems with different alignment and/or shape of the particles, the Mori-Tanaka predictions, as well as the effective field method in the form presented above, fail to satisfy the necessary symmetry conditions:

$$C^*_{ijkl} = C^*_{klij}$$

and thus they are not acceptable. This failure, in particular, suggests that the effective field method should be modified for media with more complicated internal structure. In contrast with the Mori-Tanaka approach, the possible way to improve the effective field method is clear: to introduce, for example, different effective fields for inclusions of different shapes. But this modification at the moment needs some additional study.

9 Elastic wave propagation in matrix composites

The effective field method can also be successfully applied to the solution of the homogenization problem in the dynamic case. In this section we shall find the effective dynamic characteristics of composite materials and construct the effective wave operator for a composite medium in the long wave limit approximation. The general scheme of the effective field method is developed here for this purpose.

9.1 The dynamic Green's function for an unbounded elastic medium

We consider the unbounded elastic medium with elastic moduli tensor C^0_{ijkl} and density ρ_0 without the action of body forces. The displacement field $u_i(x,t)$ in an arbitrary point x of the medium satisfies the equation of motion

$$C^0_{ijkl}\frac{\partial^2 u_l(x,t)}{\partial x_j \partial x_k} - \rho_0 \frac{\partial^2 u_i(x,t)}{\partial t^2} = 0. \tag{9.1}$$

In the case of steady-state vibrations the vector $u(x,t)$ depends upon t only through a factor $\exp(-i\omega t)$: $u_i(x,t) = u_i(x)\exp(-i\omega t)$ and for the amplitude value $u_i(x)$ we have

$$L^0_{ij}u_j(x) = 0, \quad L^0_{ik} = \nabla_j C^0_{ijkl} \nabla_l + \rho_0 \omega^2 \delta_{ik}. \tag{9.2}$$

The Green's tensor G_{ik} of the operator L^0_{ik} is defined by the equality

$$L^0_{ik} G_{kj}(x) = -\delta_{ij}\delta(x) \tag{9.3}$$

The application of Fourier transformation to both sides of (9.3) gives

$$L^0_{ik}(k,\omega)G_{kj}(k)=\delta_{ij},\quad L^0_{ik}(k,\omega)=k_jC^0_{ijkl}k_l-\rho_0\omega^2\delta_{ik} \tag{9.4}$$

It follows from here that

$$G_{ij}(k)=\left[k^2\Lambda_{ij}(\xi)-\rho_0\omega^2\delta_{ij}\right]^{-1} \tag{9.5}$$

$$\Lambda_{ij}(\xi)=C^0_{ikjl}\xi_k\xi_l,\quad \xi_i=\frac{k_i}{k},\quad k=|k|$$

The x-presentation of the function $G(k)$ has the form

$$G_{ij}(x)=\frac{1}{(2\pi)^3}\int\limits_{S_1}dS_\xi\int\limits_0^\infty\left[k^2\Lambda_{ij}(\xi)-\rho_0\omega^2\delta_{ij}\right]^{-1}\exp[-ikr(n\cdot\xi)]k^2dk \tag{9.6}$$

where S_1 is the unit sphere surface, $r=|x|$, $n_i=x_i/r$.
First of all we consider the internal integral in this expression, in which the variable change $t=kr,\quad s=(n\cdot\xi)$ is introduced:

$$J_{ij}=\int\limits_0^\infty\left[k^2\Lambda_{ij}(\xi)-\rho_0\omega^2\delta_{ij}\right]^{-1}\exp[-ikrs]k^2dk=$$

$$=\frac{1}{r}\int\limits_0^\infty\left[t^2\Lambda_{ij}(\xi)-\lambda^2\delta_{ij}\right]^{-1}\exp[-its]t^2dt \tag{9.7}$$

where: $\lambda^2=\rho_0\omega^2$.
Because of the equality

$$t^2\left[t^2\Lambda_{ij}(\xi)-\lambda^2\delta_{ij}\right]^{-1}=\Lambda^{-1}_{ij}+\lambda^2\left[t^2\Lambda_{ik}(\xi)-\lambda^2\delta_{ik}\right]^{-1}\Lambda^{-1}_{kj} \tag{9.8}$$

the integral (9.7) can be presented in the form

$$J_{ij}=\frac{1}{r}\left(J^{(0)}\delta_{ik}+J^{(1)}_{ik}\right)\Lambda^{-1}_{kj}(\xi) \tag{9.9}$$

$$J^{(0)}=\int\limits_0^\infty e^{-its}dt,\quad J^{(1)}_{ij}=\lambda^2\int\limits_0^\infty\left[t^2\Lambda_{ij}(\xi)-\lambda^2\delta_{ij}\right]^{-1}e^{-its}dt.$$

The integral $J^{(0)}$ is the generalized function

$$J^{(0)} = \frac{i}{s} + \pi\delta(s) \tag{9.10}$$

To calculate the second integral $J_{ij}^{(1)}$ let us note that the orthogonal basis of the unit vectors $e^{(k)}$ $(k = 1, 2, 3)$ exists, in which the tensor $\Lambda_{ij}(\xi)$ allows the following representation

$$\Lambda_{ij}(\xi) = \sum_{k=1}^{3} v_k^2(\xi) e_i^{(k)}(\xi) e_j^{(k)}(\xi). \tag{9.11}$$

In this basis we have

$$\left[t^2\Lambda_{ij}(\xi) - \lambda^2\delta_{ij}\right]^{-1} = \sum_{k=1}^{3} (t^2 v_k^2(\xi) - \lambda^2)^{-1} e_i^{(k)}(\xi) e_j^{(k)}(\xi). \tag{9.12}$$

Therefore, the calculation of the integral $J_{ij}^{(1)}$ is reduced to the calculation of three integrals

$$J_{ij} = \lambda^2 \sum_{k=1}^{3} \frac{e_i^{(k)}(\xi) e_j^{(k)}(\xi)}{v_k^2(\xi)} \int_0^\infty \frac{e^{-its}}{t^2 - \mu_k^2} dt, \quad \mu_k(\xi) = \frac{\lambda}{v_k(\xi)} \tag{9.13}$$

Each of them has a singular point that lies on the real axis of the complex plane, because $\mu_k(\xi)$ $(k = 1, 2, 3)$ are the real numbers. The contour of integration is chosen so that the Green's function would describe outgoing waves (the Causality Principle). Satisfying this condition, we find

$$\int_0^\infty \frac{e^{-its}}{t^2 - \mu_k^2} dt = P.V. \int_0^\infty \frac{\cos ts}{t^2 - \mu_k^2} dt + \lim_{\rho\to\infty} \int_{\gamma_\rho} \frac{\cos ts}{t^2 - \mu_k^2} dt + f(s, \mu_k) \tag{9.14}$$

Here $P.V.$ means the principal value of the integral, γ_ρ is the semi-circle with radius ρ with the center in the point μ_k, which bypasses the point μ_k in the lower half of the complex plane t, $f(s, \mu_k)$ is an odd function with argument s.
The first integral in the right-hand side of Eq. (9.14) is equal to

$$P.V. \int_0^\infty \frac{\cos ts}{t^2 - \mu_k^2} dt = -\frac{\pi}{2\mu_k} \sin \mu_k |s| \tag{9.15}$$

and the integral over the semi-circle γ_ρ gives

$$\lim_{\rho\to\infty} \int_{\gamma_\rho} \frac{\cos ts}{t^2 - \mu_k^2} dt = \frac{\pi i}{2\mu_k} \cos \mu_k s. \tag{9.16}$$

Since during integration over the unit sphere in (9.6) the term proportional to $f(s, \mu_k)$ vanishes because it is odd, then we have, up to odd terms that vanish after integration,

$$J_{ij}^{(1)} = \lambda^2 \sum_{k=1}^{3} \frac{e_i^{(k)}(\xi) e_j^{(k)}(\xi)}{v_k^2(\xi)} \cdot \frac{\pi i}{2\mu_k} \exp(i\mu_k|s|) \tag{9.17}$$

Because of the orthogonality of the basis $e^{(k)}$ this expression can be rewritten in the form

$$J_{ij}^{(1)} = \frac{i\pi\lambda}{2} \left[\sum_{k=1}^{3} \frac{e_i^{(k)}(\xi) e_m^{(k)}(\xi)}{v_k^2(\xi)} \right] \sum_{n=1}^{3} \exp\left(\frac{i\lambda|s|}{v_n(\xi)} \right) e_m^{(n)}(\xi) e_j^{(n)}(\xi) \tag{9.18}$$

According to the determination of a function with tensor arguments we have

$$\sum_{n=1}^{3} \exp\left(\frac{i\lambda|s|}{v_n(\xi)} \right) e_m^{(n)}(\xi) e_j^{(n)}(\xi) = \exp\left[i\lambda|s|\Lambda_{mj}^{-\frac{1}{2}}(\xi) \right] \tag{9.19}$$

where the tensor $\Lambda_{ij}(\xi)$ is represented in the form (9.11). From here and (9.18) we obtain finally

$$J_{ij}^{(1)} = \frac{i\pi\lambda}{2} \Lambda_{ij}^{-\frac{1}{2}}(\xi) \exp\left[i\lambda|s|\Lambda_{ij}^{-\frac{1}{2}}(\xi) \right] \tag{9.20}$$

Substituting now (9.10) into the expression (9.9) we find the internal integral

$$J_{ij} = \frac{1}{r} \left\{ \left[\frac{i}{s} + \pi\delta(s) \right] \Lambda_{ij}^{-1}(\xi) + \frac{i\pi\lambda}{2} \Lambda_{ik}^{-\frac{3}{2}}(\xi) \exp\left[i\lambda|s|\Lambda_{kj}^{-\frac{1}{2}}(\xi) \right] \right\}. \tag{9.21}$$

Let us consider the integral over the unit sphere, which corresponds to the contribution of the first term in the right-hand side of (9.21) in the expression for the tensor $G_{ij}(x)$

$$\int_{S_1} [(n \cdot \xi)\Lambda_{ij}(\xi)]^{-1} \, dS_\xi = \int_{-1}^{1} \frac{1}{s} \Psi_{ij}(s) ds \tag{9.22}$$

$$\Psi_{ij}(s) = \int_{S_1} \Lambda_{ij}^{-1}(\xi)\delta[(n \cdot \xi) - s] dS_1.$$

Because of the parity of the function $\Psi_{ij}(s)$

$$\int_{-1}^{1} \frac{1}{s} \Psi_{ij}(s) ds = 0 \tag{9.23}$$

and the expression for the Green's function $G_{ij}(x,\omega)$ takes the form

$$G_{ij}(x,\omega) = G^s_{ij}(x) + G^\omega_{ij}(x,\omega). \tag{9.24}$$

Here we have denoted

$$G^s_{ij}(x) = \frac{1}{8\pi^2 r}\int\limits_{S_1} \Lambda^{-1}_{ij}(\xi)\delta(n\cdot\xi)dS_\xi \tag{9.25}$$

$$G^\omega_{ij}(x,\omega) = \frac{i\omega}{16\pi^2}\int\limits_{S_1} \Lambda^{-\frac{3}{2}}_{ik}(\xi)\exp\left[i\omega r|n\cdot\xi|\sqrt{\rho_0}\Lambda^{-\frac{1}{2}}_{kj}(\xi)\right]dS_\xi.$$

Let us note that $G^s_{ij}(x)$ is the "static" Green's tensor, i.e. the Green's function of operator L^0 in (9.2) when $\omega = 0$.
Expanding the exponent in the integrand in the expression for $G^\omega_{ij}(x,\omega)$, we obtain the following representation of this tensor:

$$G^\omega_{ij}(x,\omega) = \frac{1}{r}\sum_{k=1}^{\infty}\frac{(i\omega r)^k}{(k-1)!}G^{(k)}_{ij}(n) \tag{9.26}$$

$$G^{(k)}_{ij}(n) = \frac{\rho_0^{\frac{k}{2}}}{16\pi^2}\int\limits_{S_1} \Lambda^{-\frac{1}{2}(k+2)}_{ij}(\xi)|n\cdot\xi|^{k-1}dS_\xi.$$

If in particular, the medium is isotropic, with Lame's constants λ_0 and μ_0, then

$$\rho^{\frac{k}{2}}\Lambda^{-k}_{ij}(\xi) = \frac{1}{v_T^{2k}}(\delta_{ij}-\xi_i\xi_j) + \frac{1}{v_L^{2k}}\xi_i\xi_j \tag{9.27}$$

where

$$v_T^2 = \frac{\mu_0}{\rho_0}, \quad v_L^2 = \frac{\lambda_0 + 2\mu_0}{\rho_0}$$

are the velocities of the shear and longitudinal wave propagation respectively.
In this case we have

$$G^s_{ij}(x) = \frac{1}{8\pi\mu_0 r}[(1+\eta^2)\delta_{ij} + (1-\eta^2)n_i n_j], \quad \eta = \frac{v_T}{v_L} \tag{9.28}$$

and the functions $G^{(k)}_{ij}(n)$ in the expansion (9.26) are determined by the expressions

$$G^{(k)}_{ij}(n) = \frac{v_T^{-(k+2)}}{4\pi\rho_0 k(k+2)}\left[(k+\eta^{k+2})\delta_{ij} + (k-1)(\eta^{k+2}-1)n_i n_j\right] \tag{9.29}$$

9.2 Scattering of elastic waves on an isolated inclusion

Let us consider an unbounded elastic medium with tensor of elastic moduli C^0 and density ρ_0, containing an inhomogeneity V of elastic characteristics C and density ρ. If the medium harmonically oscillates with frequency ω, the amplitude of its displacement vector $u(x)$ satisfies the equation of motion

$$\nabla_i[C_{ijkl}(x)\nabla_k u_l(x)] + \omega^2\rho(x)u_j(x) = 0 \tag{9.30}$$

$$C(x) = C^0 + C^1V(x), \quad \rho(x) = \rho_0 + \rho_1 V(x)$$

This system of differential equations is equivalent to the following integral equation:

$$u_i(x) = u_i^0(x) + \int_V \nabla_j G_{ik}(x-x')C^1_{jklm}\varepsilon_{lm}(x')dx' + \rho_1\omega^2\int_V G_{ik}(x-x')u_k(x')dx' \tag{9.31}$$

Here $u^0(x)$ is the "excitation" field which would have existed in a homogeneous medium with properties C^0, ρ_0 under the same acting forces; $G(x)$ is the Green's function determined in (9.24) above.
The equation for the strain tensor $\varepsilon_{ij} = \nabla_{(i}u_{j)}$ is a consequence of Eq. (9.31)

$$\varepsilon_{ij}(x) = \varepsilon^0_{ij}(x) + \int_V P_{ijkl}(x-x')C^1_{klmn}\varepsilon_{mn}(x')dx' +$$

$$+\rho_1\omega^2\int_V \nabla_{(i}G_{j)k}(x-x')u_k(x')dx', \quad P_{ijkl}(x) = \nabla_{j)}\nabla_{(l}G_{k)(i}(x) \tag{9.32}$$

If the length of the incident wave substantially exceeds the characteristic linear dimension of the typical inclusion, Eqs. (9.31) and (9.32) can be solved approximately in the long wave limit. To this end we must retain only the main terms in the series (9.26). In turn, keeping only the essential terms in the real and imaginary parts of the latter, we have

$$G(x) = G^s(x) + i\omega G^{(1)} - i\omega^3|x|^2G^{(3)}(n) \tag{9.33}$$

It should be emphasized that ω can be considered here as a small parameter. In this case the solution of Eqs. (9.31) and (9.32) can be constructed as an expansion with respect to ω.

The consequence of Eq. (9.33) is the following expression for the kernel $P(x)$

$$P(x,\omega) = P^s(x) + i\omega^3 H, \quad P^s_{ijkl}(x) = \nabla_{l)}\nabla_{(j}G^s_{i)(k}(x)$$

$$H_{ijkl} = \frac{\rho_0^{3/2}}{16\pi^2}\int\limits_{S_1} \xi_{(i}\Lambda^{-5/2}_{j)(k}(\xi)\xi_{l)}dS_\xi \tag{9.34}$$

Note that H is a constant tensor.
Let us find the solution of Eqs. (9.31) and (9.32) in the form

$$u(x) = u^R(x) + i\omega^3 u^\omega(x), \quad \varepsilon(x) = \varepsilon^R(x) + i\omega^3\varepsilon^\omega(x) \tag{9.35}$$

Substituting Eq. (9.35) in Eqs. (9.31) and (9.32) and comparing the terms of the same order with respect to ω we will obtain the following equations for the components of the displacement vector u :

$$u^R(x) = u^0(x), \quad u^\omega_i(x) = \rho_1 G^{(1)}_{ik}\int\limits_V u^0_k(x)dx \tag{9.36}$$

and the integral equations for the components of the strain tensor $\varepsilon(x)$

$$\varepsilon^R_{ij}(x) - \int\limits_V P^S_{ijkl}(x-x')C^1_{klmn}\varepsilon^R_{mn}(x')dx' = \varepsilon^0_{ij} \tag{9.37}$$

$$\varepsilon^\omega_{ij}(x) - \int\limits_V P^S_{ijkl}(x-x')C^1_{klmn}\varepsilon^\omega_{mn}(x')dx' = -H_{ijkl}C^1_{klmn}\int\limits_V \varepsilon^R_{mn}(x)dx \tag{9.38}$$

Eq. (9.37) is identical to the integral equation considered above for the solution of the static one-particle problem. In the long wave approximation the oscillation of the fields u^0 and ε^0 in the region V can be neglected. If this region is ellipsoidal with semi-axes a_1, a_2, a_3 the solution of Eq. (9.37) has the form

$$\varepsilon^R = A^0\varepsilon^0 \tag{9.39}$$

where the tensor $A^0 = A^0(a)$ coincides with A in (2.66).
Substituting Eq. (9.39) into the right-hand side of Eq. (9.38) we obtain

$$\varepsilon^\omega = -\Lambda^\omega\varepsilon^0, \quad \Lambda^\omega = vA^0HC^1A^0 \tag{9.40}$$

where $v = \frac{4}{3}a_1a_2a_3$ is the volume of the ellipsoid.

Since $G^{(1)}$ is constant, the final result reads

$$\varepsilon_{ij}(x) = \Lambda_{ijkl}(a,\omega)\varepsilon^0_{kl}(x), \quad u_i(x) = \lambda_{ij}(a,\omega)u^0_j(x)$$

$$\Lambda = \Lambda^0 - i\omega^3\Lambda^\omega, \quad \lambda_{ij} = \delta_{ij} + i\omega^3 v\rho_1 G^{(1)}_{ij}. \tag{9.42}$$

Substituting these relations into the right-hand side of Eqs. (9.31) and (9.32) one can find the wave field outside the inclusion. Hence, Eq. (9.42) gives the full solution of the long wave scattering problem for an elastic homogeneous medium containing a single inclusion.

9.3 Effective wave operator for the medium with a random set of inclusions

Let us consider an unbounded medium containing a uniform random set of ellipsoidal inclusions. As before $V(x)$ is the characteristic function of the region occupied by inclusions. For harmonic oscillation, the amplitudes of the displacement $u(x)$ and strain $\varepsilon(x)$ fields in the medium satisfy equations similar to Eqs. (9.31) and (9.32):

$$u_i(x) = u^0_i(x) + \int \nabla_j G_{ik}(x-x')C^1_{kjlm}\varepsilon_{lm}(x')V(x')dx' + \tag{9.43}$$

$$+\rho_1\omega^2 \int G_{ik}(x-x')u_k(x')V(x')dx'$$

$$\varepsilon_{ij}(x) = \varepsilon^0_{ij}(x) + \int P_{ijkl}(x-x')C^1_{klmn}\varepsilon_{mn}(x')V(x')dx' + \tag{9.44}$$

$$+\rho_1\omega^2 \int \nabla_{(i}G_{j)k}(x-x')u_k(x')V(x')dx'.$$

The local displacement, $u^*(x)$, and the strain, $\varepsilon^*(x)$, are represented in the form

$$u^*_i(x) = u^0_i(x) + \int \nabla_j G_{ik}(x-x')C^1_{kjlm}\varepsilon_{lm}(x')V(x;x')dx' +$$

$$+\rho_1\omega^2 \int G_{ik}(x-x')u_k(x')V(x;x')dx', \quad x \in V \tag{9.45}$$

$$\varepsilon_{ij}^*(x) = \varepsilon_{ij}^0(x) + \int P_{ijkl}(x-x')C_{klmn}^1 \varepsilon_{mn}(x')V(x;x')dx' +$$

$$+\rho_1\omega^2 \int \nabla_{(i}G_{j)k}(x-x')u_k(x')V(x;x')dx', \quad x \in V \tag{9.46}$$

where the function $V(x;x')$ is defined by Eq. (5.1).
In the long wave approximation the fields $u^*(x)$ and $\varepsilon^*(x)$ can be considered as constant in each region $V_i, i = 1,2,3,...$, (the main hypothesis of the effective field method). Then the displacement and strain fields inside each inclusion are expressed in terms of the fields $u^*(x)$ and $\varepsilon^*(x)$, according to Eq. (9.42) obtained from the solution of the one-particle problem

$$u_i(x) = \lambda_{ij}(\omega,a)u_j^*(x), \quad \varepsilon_{ij}(x) = \Lambda_{ijkl}(\omega,a)\varepsilon_{kl}^*(x). \tag{9.47}$$

The functions $\lambda(x)$ and $\Lambda(x)$ here coincide with $\lambda(a_k)$ and $\Lambda(a_k)$ defined by Eq. (9.42), when $x \in V_i$.
Substituting Eq. (9.47) into the right-hand side of Eqs. (9.45) and (9.46) we obtain the system of self-consistent equations governing the fields $u^*(x)$ and $\varepsilon^*(x)$

$$u_i^*(x) = u_i^0(x) + \int \nabla_j G_{ik}(x-x')C_{kjlm}^1 \Lambda_{lmrs}(x')\varepsilon_{rs}^*(x')V(x;x')dx' +$$

$$+\rho_1\omega^2 \int G_{ik}(x-x')\lambda_{kr}(x')u_r^*(x')V(x;x')dx', \quad x \in V \tag{9.48}$$

$$\varepsilon_{ij}^*(x) = \varepsilon_{ij}^0(x) + \int P_{ijkl}(x-x')C_{klmn}^1 \Lambda_{mnrs}(x')\varepsilon_{rs}^*(x')V(x;x')dx' +$$

$$+\rho_1\omega^2 \int \nabla_{(i}G_{j)k}(x-x')\lambda_{kr}(x')u_r^*(x')V(x;x')dx', \quad x \in V. \tag{9.49}$$

If the solution of these equations is known, the fields $u(x)$ and $\varepsilon(x)$ in the composite medium can be found from Eqs. (9.43) and (9.44). Thus $u^*(x)$ and $\varepsilon^*(x)$ are the principal unknowns of the problem.
After the conditional averaging of Eqs. (9.48) and (9.49) and taking into account the hypotheses of the effective field method one obtains the expressions for the conditional means:

$$\mathcal{U}_i^*(x) = u_i^0(x) + \int [\nabla_j G_{ik}(x-x')C_{jkmn}^\Lambda \mathcal{E}_{mn}^*(x') +$$

$$+\omega^2 G_{ik}(x-x')\rho^\lambda_{km}\mathcal{U}^*_m(x')]\Psi(x-x')dx' \tag{9.50}$$

$$\mathcal{E}^*_{ij}(x)=\varepsilon^0_{ij}(x)+\int[P_{ijkl}(x-x')C^\Lambda_{klmn}\mathcal{E}^*_{mn}(x')+$$

$$+\omega^2\nabla_{(j}G_{i)k}(x-x')\rho^\lambda_{km}\mathcal{U}^*_m(x')]\Psi(x-x')dx' \tag{9.51}$$

Here

$$\mathcal{U}^*_i(x)=\langle u^*_i(x)|x\rangle\,,\quad \mathcal{E}^*_{ij}(x)=\left\langle\varepsilon^*_{ij}(x)|x\right\rangle,$$

C^Λ and ρ^λ are the constant tensors defined by the expressions

$$C^\Lambda=\left\langle C^1(x)\Lambda(x)V(x)\right\rangle=n_0\left\langle vC^1(x)\Lambda(x)\right\rangle$$

$$\rho^\Lambda=\rho_1\left\langle\lambda(x)V(x)\right\rangle=n_0\rho_1\left\langle v\lambda(x)\right\rangle. \tag{9.52}$$

The function

$$\Psi(x-x')=\frac{\langle V(x;x')|x\rangle}{\langle V(x)\rangle}$$

was considered earlier.
The averaging of Eqs. (9.43) and (9.44) gives

$$\mathcal{U}_i(x)=u^0_i(x)+C^\Lambda_{jkmn}\int\nabla_j G_{ik}(x-x')\mathcal{E}^*_{mn}(x')dx'+$$

$$+\omega^2\rho^\lambda_{km}\int G_{ik}(x-x')\mathcal{U}^*_m(x')dx' \tag{9.53}$$

$$\mathcal{E}_{ij}(x)=\varepsilon^0_{ij}(x)+C^\Lambda_{klmn}\int P_{ijkl}(x-x')\mathcal{E}^*_{mn}(x')dx'+$$

$$+\omega^2\rho^\lambda_{km}\int\nabla_{(j}G_{i)k}(x-x')\mathcal{U}^*_m(x')dx' \tag{9.54}$$

where

$$\mathcal{U}_i(x)=\langle u_i(x)\rangle\,,\quad \mathcal{E}_{ij}(x)=\langle\varepsilon_{ij}(x)\rangle$$

Excluding the exiting fields $u_i^0(x)$ and $\varepsilon_{ij}^0(x)$ from Eqs. (9.50), (9.51), (9.53) and (9.54) we obtain

$$\mathcal{U}_i^*(x) = \mathcal{U}_i(x) - \int [\nabla_j G_{ik}(x-x') C_{jkmn}^{\lambda} \mathcal{E}_{mn}^*(x') +$$

$$+\omega^2 G_{ik}(x-x') \rho_{km}^{\lambda} \mathcal{U}_m^*(x')] \Phi(x-x') dx' \tag{9.55}$$

$$\mathcal{E}_{ij}^*(x) = \mathcal{E}_{ij}(x) - \int [P_{ijkl}(x-x') C_{klmn}^{\Lambda} \mathcal{E}_{mn}^*(x') dx' +$$

$$+\omega^2 \nabla_{(j} G_{i)k}(x-x') \rho_{km}^{\lambda} \mathcal{U}_m^*(x')] \Phi(x-x') dx' \tag{9.56}$$

$$\Phi(x) = 1 - \Psi(x)$$

Eqs. (9.55) and (9.56) are equations of convolution type with respect to $\mathcal{U}^*$ and $\mathcal{E}^*$. The Fourier transform reduces Eqs. (9.55) and (9.56) to the algebraic equations

$$\mathcal{U}_i^*(k) = \mathcal{U}_i(k) - T_{ikl}(k) \mathcal{E}_{kl}^*(k) - t_{ik}(k) \mathcal{U}_k^*(k)$$

$$\mathcal{E}_{ij}^*(k) = \mathcal{E}_{ij}(k) - \Pi_{ijkl}(k) \mathcal{E}_{kl}^*(k) - \pi_{ijk}(k) \mathcal{U}_k^*(k) \tag{9.57}$$

Here the same notations for the Fourier transform of the functions are used, but with argument k instead of x, and we have denoted

$$T_{ikl}(k) = \left[\int \nabla_j G_{ir}(x) \Phi(x) \exp(ik \cdot x) dx \right] C_{jrkl}^{\Lambda},$$

$$t_{ik}(k) = \omega^2 \left[\int G_{ir}(x) \Phi(x) \exp(ik \cdot x) dx \right] \rho_{rk}^{\lambda},$$

$$\Pi_{ijkl}(k) = \left[\int P_{ijmn}(x-x') \Phi(x) \exp(ik \cdot x) dx \right] C_{mnkl}^{\Lambda} \tag{9.58}$$

$$\pi_{ijk}(k) = \omega^2 \left[\int \nabla_{(j} G_{i)r}(x) \Phi(x) \exp(ik \cdot x) dx \right] \rho_{rk}^{\lambda}$$

For the statistically isotropic set of inclusions, $\Phi(x)$ is the smooth function rapidly going to zero outside the domain, having the order of l that is the correlation radius of the random set of inhomogeneities. Let $\mathcal{U}^*(x)$ and $\mathcal{E}^*(x)$ be in the long wave limit also smooth functions, so the supports of these functions are concentrated in the domain of k-space given by the condition $|kl| \ll 1$. In this case we can expand the function $\exp(ik \cdot x)$ in the integrands of expressions (9.58) in series leaving only the higher order terms in this expansion

$$\exp(ik \cdot x) \cong 1 + ik_i x_i - \frac{1}{2} k_i k_j x_i x_j \tag{9.59}$$

Now we have to calculate the integrals in (9.58) taking into account the expansion (9.58) and the main terms of the function $G_{ik}(x,\omega)$ in (9.33). Substituting this result in the system (9.57) and solving it with respect to $\mathcal{U}^*(k)$ and $\mathcal{E}^*(k)$ we obtain with the exactness of the main terms with respect to ω in the real and imaginary parts:

$$\mathcal{U}_i^*(k) = d_{ij}(\omega)\mathcal{U}_j(k), \quad \mathcal{E}_{ij}^*(k) = D_{ijkl}(k,\omega)(-ik_l)\mathcal{U}_k(k) \tag{9.60}$$

$$d_{ij}(\omega) = \delta_{ij} - i\omega^3 p\rho_1 G_{ij}^{(1)} J, \quad J = \int \Phi(x)dx,$$

$$D(k,\omega) = D^0\{I - i\omega^3(n_0 P^0 \langle v^2 C^1 A^0 H C^R \rangle - JHC^R) -$$

$$-l^2[P^1 \cdot (ik \otimes ik)]C^R\}, \quad D^0 = (I - n_0 P^0 \langle v C^1 A^0 \rangle)^{-1}$$

$$C^R(a) = C^1 A^0(a) D^0, \quad C^R = n_0 \langle v C^1 A^0 \rangle D^0, \quad P_{ijkl}^0 = \int P_{ijkl}^s(x)\Phi(x)dx,$$

$$P_{ijklmn}^1 = \int\limits_{S_1} P_{ijkl}^s(x)\Phi(x) n_m n_n dS_n, \quad n = \frac{k}{|k|}, \quad l^2 = \int\limits_0^\infty r\Phi(r)dr.$$

The expression for $\mathcal{U}(k)$ follows from Eqs. (9.43) and (9.47); it has the form

$$\mathcal{U}_i(k) = u_i^0(k) + (-ik_k)G_{ij}(k)C_{jklm}^\Lambda \mathcal{E}_{lm}^*(k) - \omega^2 G_{ij}(k)\rho_{jk}^\lambda \mathcal{U}_k^*(k). \tag{9.61}$$

After substituting here Eq. (9.60) for $\mathcal{U}_i^*(k)$ and $\mathcal{E}_{ij}^*(k)$ we obtain

$$u_i^0(k) = \mathcal{U}_i(k) - (-ik_k)G_{ij}(k)C_{jklm}^{\Lambda}D_{lmrs}(-ik_r)\mathcal{U}_s(k) - \omega^2 G_{ij}(k)\rho_{jk}^{\lambda}d_{kl}\mathcal{U}_l(k). \tag{9.62}$$

Let us multiply both sides of this equation by the tensor

$$L_{ik}^0(k,\omega) = C_{ijkl}^0 k_j k_l - \rho_0\delta_{ik}$$

and take into account the obvious relations

$$L_{ik}^0(k,\omega)u_k^0(k) = 0, \quad L_{ik}^0(k,\omega)G_{kj}(k,\omega) = \delta_{ij}. \tag{9.63}$$

As a result we obtain the following equation for $\mathcal{U}(k)$ in the (k,ω)-representation

$$L_{ik}^*(k,\omega)\mathcal{U}_k(k) = 0, \quad L_{ik}^*(k,\omega) = k_k C_{ijkl}^* k_l - \omega^2\rho_{ik}^*(k,\omega) \tag{9.64}$$

$$C^*(k,\omega) = C^s - l^2 C^R[P^1\cdot(ik\otimes ik)]C^R - i\omega^3[n_0^2\left\langle v^2C^RHC^R\right\rangle - JC^RHC^R]$$

$$C^s = C^0 + C^R, \quad \rho_{ij}^* = \rho_s\delta_{ij} + i\omega^3 p\rho_1 f G_{ij}^{(1)}$$

$$\rho_s = \rho_0 + p\rho_1, \quad f = \langle v\rangle - pJ = \langle v\rangle(1 - n_0 J) \tag{9.65}$$

Thus the averaged wave field $\mathcal{U}(x)$ satisfies the equation

$$(L^*\mathcal{U})(x) = 0 \tag{9.66}$$

where the operator L^* acts on $\mathcal{U}(x)$ according to the rule

$$(L^*\mathcal{U})(x) = \frac{1}{(2\pi)^3}\int L^*(k)\mathcal{U}(k)\exp(-ik\cdot x)dk \tag{9.67}$$

The operator L^* can be called the effective wave operator for the composite medium. It follows from Eqs. (9.64) and (9.66) that in the x-space the latter has the form

$$L_{ik}^* = -\nabla_j C_{ijkl}^*\nabla_l - \rho_{ik}^*\omega^2 \tag{9.68}$$

This operator describes wave propagation through a certain homogeneous medium. It should be emphasized that C^* as well as ρ^* in Eq. (9.68) are also operators.

C^* is the sum of the constant tensor $C^s - i\omega^3 C^\omega$ and a differential operator of the second order:

$$C^* = C^s - i\omega^3 C^\omega + l^2 C^R[P^1 \cdot (\nabla \otimes \nabla)]C^R$$

$$C^\omega = n_0^2 \left\langle vC^R H C^R \right\rangle - JC^R H C^R. \tag{9.69}$$

Thus the equivalent medium exhibits temporal and spatial dispersion. The speeds of elastic waves in such a medium are determined by the real parts of C^* and ρ^*. The imaginary parts of these tensors describe the attenuation of waves because of scattering on the inclusions.

9.4 Green's function of the effective wave operator

Let us consider the general formulae obtained in the previous section in the case of composites with isotropic constituents. We assume for simplicity that the inclusions are identical spheres with radius a and their elastic properties and densities are the same. Thus only the spatial distribution of the inclusions is random. In this case the operator L^* has the usual form for isotropic media:

$$L^*_{ik} = \left[\left(K^* - \frac{2}{3}\mu^*\right) n_i n_k + \mu^* \delta_{ik}\right] k^2 - \rho^* \omega^2 \delta_{ik}. \tag{9.70}$$

Here K^* and μ^* are the bulk and shear moduli, ρ^* is the density of the composite material,

$$K^* = K_s + p^2 l^2 k^2 K_l - i\omega^3 pf K_\omega, \quad K_s = K_0 + pK_R$$

$$\mu^* = \mu_s + p^2 l^2 k^2 \mu_l - i\omega^3 pf \mu_\omega, \quad \mu_s = \mu_0 + p\mu_R$$

$$K_R = \left[\frac{1}{K_1} + 3(1-p)P_1^0\right]^{-1}, \quad \mu_R = \left[\frac{1}{\mu_1} + 2(1-p)P_2^0\right]^{-1}$$

$$K_l = \frac{4\mu_R}{105\mu_0}\left[14K_R + \frac{1}{3}\mu_R(3+4\eta^2)\right], \quad \mu_l = \frac{\mu_R^2}{105\mu_0}(3+4\eta^2)$$

$$P_1^0 = \frac{3-4\eta^2}{9K_0}, \quad P_2^0 = \frac{3+2\eta^2}{15\mu_0}, \quad K_\omega = 9K_R^2 H_1, \quad \mu_\omega = 2\mu_R^2 H_2$$

$$H_1 = \frac{\eta^5}{36\pi\rho_0 v_T^5}, \quad H_2 = \frac{3+2\eta^5}{60\pi\rho_0 v_T^5}, \quad \eta = \frac{v_T}{v_L}$$

$$\rho^* = \rho_s + i\omega^3 pf\rho_\omega, \quad \rho_\omega = \frac{\rho_1(2+\eta^3)}{12\pi\rho_0 v_T^3}. \tag{9.71}$$

Let us introduce the projection operators Π and Θ

$$\Pi_{ij}(k) = n_i n_j, \quad \Theta_{ij}(k) = \delta_{ij} - n_i n_j, \quad n = k/|k|. \tag{9.72}$$

These operators allow to separate the longitudinal (L) from the transverse (T) components of the wave field. The operator L^* can be represented as the sum of two orthogonal components:

$$L^*(k,\omega) = L_L^*(k,\omega)\Pi(k) + L_T^*(k,\omega)\Theta(k)$$

$$L_L^*(k,\omega) = k^2\left(K^* + \frac{4}{3}\mu^*\right) - \rho^*\omega^2, \quad L_T^*(k,\omega) = k^2\mu^* - \omega^2\rho^* \tag{9.73}$$

We construct now the Green's tensor G^* of this operator. In the (k,ω)-representation the tensor $G^*(k,\omega)$ takes the form

$$G^*(k,\omega) = [L^*(k,\omega)]^{-1} = \frac{1}{L_L^*(k,\omega)}\Pi + \frac{1}{L_T^*(k,\omega)}\Theta. \tag{9.74}$$

Let us consider the longitudinal part of this tensor in detail:

$$G_{ij}^{*L}(k,\omega) = \frac{1}{L_L^*(k,\omega)} \cdot \frac{k_i k_j}{k^2}$$

Performing the inverse Fourier transform, we have

$$G_{ij}^{*L}(x,\omega) = \nabla_i\nabla_j\left[\frac{1}{(2\pi)^3}\int \frac{\exp(-ik\cdot x)dk}{k^2(-k^2\kappa^* + \omega^2\rho^*)}\right]. \tag{9.75}$$

Here

$$\kappa^* = \kappa_s + (kpl)^2\kappa_l - i\omega^3 pf\kappa_\omega \tag{9.76}$$

$$\kappa_s = K_s + \frac{4}{3}\mu_s, \quad \kappa_l = K_l + \frac{4}{3}\mu_l, \quad \kappa_\omega = K_\omega + \frac{4}{3}\mu_\omega.$$

To calculate the integral in (9.75) we introduce the spherical system of coordinates (r, θ, φ) with the polar axis directed along the vector $\mathbf{r}$. It gives

$$G_{ij}^{*L}(x,\omega) = \frac{1}{(2\pi)^3}\nabla_i\nabla_j \left\{ \int_0^\infty \frac{dk}{-k^2\kappa^* + \omega^2\rho^*} \int_{S_1} \exp[-ikr(n\cdot e)]dS_n \right\} \tag{9.77}$$

where $e = x/|x|$, and the integral in the right-hand side of this expression is over the unit sphere. After calculating the latter, Eq. (9.77) transforms to the following one-dimensional integral

$$G_{ij}^{*L}(x,\omega) = \frac{1}{2\pi^2}\nabla_i\nabla_j \left[\frac{1}{ir} \int_{-\infty}^{\infty} \frac{\exp(ikr)}{k(-k^2\kappa^* + \omega^2\rho^*)} dk \right]$$

that can be transformed to:

$$\int_{-\infty}^{\infty} \frac{\exp(ikr)}{k(-k^2\kappa^* + \omega^2\rho^*)} dk = -\frac{1}{l^2\kappa_l(k_1^2 - k_3^2)} \left[\left(\frac{1}{k_3^2} - \frac{1}{k_1^2} \right) \int_{-\infty}^{\infty} \frac{\exp(ikr)dk}{k} + \right.$$

$$\left. + \frac{1}{k_1^2} \int_{-\infty}^{\infty} \frac{k\exp(ikr)}{k^2 - k_1^2} dk - \frac{1}{k_3^2} \int_{-\infty}^{\infty} \frac{k\exp(ikr)}{k^2 - k_3^2} dk \right] \tag{9.78}$$

Here k_1 and k_3 are the roots of the equation

$$k^4 + \frac{1}{pl^2\kappa_l}(\kappa_s - i\omega^3 pf\kappa_\omega)k^2 - \frac{\omega^2}{(pl)^2\kappa_l}(\rho_s + i\omega^3 pf\rho_\omega) = 0$$

that lie in the upper half of the complex plane k. With the accepted accuracy we can write

$$k_1 = \omega\sqrt{\frac{\rho_s}{\kappa_s}} \left[1 + \frac{1}{2} i\omega^3 pf \left(\frac{\rho_\omega}{\rho_s} + \frac{\kappa_\omega}{\kappa_s} \right) \right]$$

$$k_3 = \frac{f\omega^3\kappa_\omega}{2l\sqrt{\kappa_s\kappa_l}} + \frac{i}{lp}\sqrt{\frac{\kappa_s}{\kappa_l}} \tag{9.79}$$

The first integral on the right-hand side of (9.78) is the Fourier transform of the generalized function k^{-1} and is equal to πi. The two remaining integrals are calculated with the theory of residuals. We obtain the result:

$$\frac{1}{ir} \int_{-\infty}^{\infty} \frac{\exp(ikr)}{kL_L^*(k,\omega)} dk = \frac{\pi}{l^2\kappa_l(k_1^2 - k_3^2)} \left[\frac{1 - e^{ik_3 r}}{k_3^2 r} - \frac{1 - e^{ik_1 r}}{k_1^2 r} \right].$$

The finite expression for the function $G_{ij}^{*L}(x,\omega)$ takes the form

$$G_{ij}^{*L}(x,\omega) = \frac{1}{4\pi}\nabla_i\nabla_j\left[\frac{1}{\rho_s\omega^2 r}(1-e^{ik_1 r}) + \frac{(pl)^2\kappa_l}{\kappa_s^2 r}(1-e^{ik_3 r})\right]. \tag{9.81}$$

Omitting the analogous intermediate calculations we present here the expression for the transversal component of the Green's tensor $G_{ij}^{*L}(x,\omega)$

$$G_{ij}^{*T}(x,\omega) = \frac{1}{4\pi\mu_s}\left\{\frac{1}{r}\left(e^{ik_2 r} - e^{ik_4 r}\right)\delta_{ij} - \right.$$

$$\left. -\nabla_i\nabla_j\left[\frac{\mu_s}{\rho_s\omega^2 r}(1-e^{ik_2 r}) + \frac{l^2\mu_l}{\mu_s r}(1-e^{ik_4 r})\right]\right\}. \tag{9.81}$$

Here k_2 and k_4 are the roots of the equation

$$k^4 + \frac{1}{(pl)^2\mu_l}(\mu_s - i\omega^3 pf\mu_\omega)k^2 - \frac{\omega^2}{(pl)^2\mu_l}(\rho_s + i\omega^3 pf\rho_\omega) = 0$$

that lie in the upper half of the plane k. The expressions for these roots are

$$k_2 = \omega\sqrt{\frac{\rho_s}{\mu_s}}\left[1 + \frac{1}{2}i\omega^3 pf\left(\frac{\rho_\omega}{\rho_s} + \frac{\mu_\omega}{\mu_s}\right)\right]$$

$$k_3 = \frac{f\omega^3\mu_\omega}{2l\sqrt{\mu_s\mu_l}} + \frac{i}{lp}\sqrt{\frac{\mu_s}{\mu_l}} \tag{9.82}$$

Hence, the full Green's tensor is determined by the expression

$$G_{ij}^{*}(x,\omega) = \frac{1}{4\pi\mu_s}\left[\frac{e^{ik_2 r}}{r}\delta_{ij} - \frac{\mu_s}{\rho_s\omega^2}\nabla_i\nabla_j\left(\frac{e^{ik_1 r}}{r} - \frac{e^{ik_2 r}}{r}\right)\right] -$$

$$-\frac{1}{4\pi\mu_s}\left\{\frac{e^{ik_4 r}}{r}\delta_{ij} - (pl)^2\nabla_i\nabla_j\left[\frac{\mu_l}{\mu_s r}\left(1-e^{ik_4 r}\right) - \frac{\kappa_l\mu_s}{\kappa_s^2 r}\left(1-e^{ik_3 r}\right)\right]\right\}, \quad r = |x| \tag{9.83}$$

If ω tends to zero in expression (9.83), we obtain the static Green function G_s^* of the homogeneous medium that is equivalent to the composite material under study:

$$G_{(s)ij}^{*}(x,\omega) = \frac{1}{4\pi\mu_s r}\left[1 - \exp\left(-\frac{r}{pl}\sqrt{\frac{\mu_s}{\mu_l}}\right)\right]\delta_{ij} -$$

$$-\frac{1}{8\pi\mu_s}\nabla_i\nabla_j\left\{\left(1-\frac{\mu_s}{\kappa_s}\right)r+\frac{2l^2}{\mu_s r}\left[1-\exp\left(-\frac{r}{pl}\sqrt{\frac{\mu_s}{\mu_l}}\right)\right]\right\}+$$

$$+\frac{1}{8\pi\mu_s}\nabla_i\nabla_j\left\{\frac{2(pl)^2\kappa_l\mu_s}{\kappa_s^2 r}\left[1-\exp\left(-\frac{r}{pl}\sqrt{\frac{\kappa_s}{\kappa_l}}\right)\right]\right\} \tag{9.84}$$

Contrary to the classical Green's function for a homogeneous elastic medium this function has no singularity at $r = 0$. Boundedness at zero is a characteristic property of so-called quasicontinua and other nonlocal models of media with micro-structure. If $r \to \infty$ the right-hand side of Eq. (9.84) possesses the classical asymptotics r^{-1} , as well as terms of order r^{-3} and terms falling off exponentially at distances of the order of the correlation radius l of the random set of inclusions. If the correlation radius l is much smaller than the characteristic scale of the varying external field, then Eq. (9.84) turns into the classical Green's function for a homogeneous medium with the effective elastic moduli K_s, μ_s

$$G^*_{(s)ij} = \frac{1}{8\pi\mu_s}\left[\frac{2}{r}\delta_{ij} - \left(1-\frac{\mu_s}{\kappa_s}\right)\nabla_i\nabla_j r\right]. \tag{9.85}$$

This is the main term of the asymptotics of the right-hand side of Eq. (9.78) as $l \to 0$.

9.5 Velocities and attenuation factors of elastic waves in matrix composites

Let us analyze Eq. (9.76) for the dynamic Green tensor. Obviously its right-hand side describes two types of waves that propagate out of a point source. The first of them, with the wave numbers k_1 and k_2, represents the attenuating waves with attenuating factors that are proportional to $(\omega l)^4$. Waves of the second type, characterized by the wave numbers k_3 and k_4, attenuate much more rapidly than the first ones (their attenuation is over a distance of the order of the correlation radius l). Thus in a number of cases we can disregard waves of the second type and assume that the effective dynamic Green tensor G^* is defined by the expression

$$G^*_{ij}(x,\omega) = \frac{1}{4\pi\mu_s}\left[\frac{e^{ik_2r}}{r}\delta_{ij} + \frac{\mu_s}{\rho_s\omega^2}\nabla_i\nabla_j\left(\frac{e^{ik_2r}}{r}-\frac{e^{ik_1r}}{r}\right)\right] \tag{9.86}$$

i.e., it has the same form as the Green tensor of a homogeneous isotropic medium with elastic moduli K_s and μ_s and density ρ_s. The wave numbers k_1 and k_2 are complex quantities in this case. Their real parts define the effective propagation velocities

$$v_L^* = \sqrt{\frac{\kappa_s}{\rho_s}}, \quad v_T^* = \sqrt{\frac{\mu_s}{\rho_s}} \tag{9.87}$$

of longitudinal and transverse waves in a medium with inclusions. It follows from Eq. (9.87), that v_L^* and v_T^* are independent on the frequency and, hence, there is no velocity dispersion in this case. This stems from the fact that in the real parts of all preceding expressions the terms of order ω^2 were assumed small in comparison to unity and were discarded. A consideration of these terms in the suggested scheme only increases the technical difficulties without yielding a considerable improvement of the results obtained.

The imaginary parts of the wave numbers k_1 and k_2 are the per-unit-length attenuation factors. Using the formulae for the coefficients in Eq. (9.77) we can write the expressions for $\gamma_L = \mathrm{Im}(k_1)$ and $\gamma_T = \mathrm{Im}(k_2)$ as follows

$$\gamma_L = \frac{pf}{24\pi\rho_0\rho_s}\left(\frac{\omega}{v_L^*}\right)^4 \left\{\frac{v_L^*}{v_L^5}\left[K_R^2 + \frac{8}{15}\mu_R^2\left(\frac{1}{\eta^5}+\frac{2}{3}\right)\right] + \left(\frac{v_L^*}{v_L}\right)^3 \rho_1^2\left(1+\frac{2}{\eta^3}\right)\right\}$$

$$\gamma_T = \frac{pf}{24\pi\rho_0\rho_s}\left(\frac{\omega}{v_T^*}\right)^4 \left[\frac{2v_T^*}{5v_T^5}\mu_R^2(3+2\eta^5) + \left(\frac{v_T^*}{v_T}\right)^3 \rho_1^2\left(2+\eta^3\right)\right] \tag{9.88}$$

In accordance with their physical meaning the attenuation factors are positive. Consequently the multiplier f in the preceding expressions should satisfy the condition

$$f = \langle v \rangle - 4\pi p \int_0^\infty \Phi(r) r^2 dr \geq 0 \tag{9.89}$$

This imposes constraints on the volume concentration p for which the resultant formulae remain physically consistent. For example, for a function $\Phi(r)$ of the form

$$\Phi(r) = \begin{cases} 1 & r \leq 2a \\ 0 & r > 2a \end{cases}$$

γ_L^* and γ_T^* are positive only for $p \leq 1/8$.

We must emphasize that the closure condition used above, which determined the structure of the multiplier f in Eq. (9.88), is hardly valid for high concentrations of inclusions. It is obvious that correlation in spatial locations of inclusions increases with increasing p. Hence sufficiently simple approximations of $\Phi(r)$ are possible only for small p. A construction of this function for high concentrations of inclusions presents considerable difficulties. The solution of the so-called Perkus-Yevick equation provides one of the possible approximations of $\Phi(r)$. The function $\Phi(r)$ based on this solution leads to the following expression for the multiplier f in Eqs. (9.88)

$$f = \langle v\rangle \frac{(1-p)^4}{(1+2p)^2} \tag{9.90}$$

It remains positive in the whole range of possible p.
Let identical inclusions compose a regular lattice in a homogeneous matrix. In this case the function $\Phi(r)$ is periodic with zero mean value over the cell of the lattice. The exception is the cell with the center in the origin where the function $\Phi(x)$ is equal to unity. Hence the integral in the expression for f equals the volume of the cell

$$\int \Phi(x)dx = \frac{1}{n_0} \tag{9.91}$$

Thus $f = 0$ in this case. It corresponds to the well-known fact that a long wave propagating through periodical structures is free from attenuation.
If the volume concentration of the inclusions is so small that their interaction can be disregarded, we should discard terms of the order greater than p in Eqs. (9.88). In this case the formulae for γ_L and γ_T become

$$\gamma_L = \frac{1}{2}n_0Q_L, \quad \gamma_T = \frac{1}{2}n_0Q_T \tag{9.92}$$

$$Q_L = \frac{4\pi a^2}{9}\left(\frac{\omega a}{v_L}\right)^4\left\{\frac{1}{(\rho_0 v_L^2)^2}\left[K_R^2 + \frac{8}{15}\mu_R^2\left(\frac{1}{\eta^5}+\frac{2}{3}\right)\right] + \left(\frac{\rho_1}{\rho_0}\right)^2\left(1+\frac{2}{\eta^3}\right)\right\}$$

$$Q_T = \frac{4\pi a^2}{9}\left(\frac{\omega a}{v_T}\right)^4\left[\frac{\mu_R^2}{15(\rho_0 v_T^2)^2}(3+2\eta^5) + \frac{1}{3}\left(\frac{\rho_1}{\rho_0}\right)^2(2+\eta^3)\right]$$

Here Q_L and Q_T are the well-known expressions for the full scattering cross-sections of longitudinal and shear waves by a single spherical inclusion of radius a. Thus Eqs.(9.92) have the exact values of the attenuation factors of long waves for noninteracting inclusions.

10 The piezoelectric composite materials

So far we dealt with the purely elastic composite material. However, if the constituents in the composite exhibit electromechanical coupling (being piezoelectric, for example), an external elastic field can produce a mechanical deformation which generates an internal stress field and vice versa. Both of these fields can influence the macroscopic response of the composite material. Therefore, the coupling effects must be taken into account when estimating the overall properties of composites with piezoactive components. The effective field method can be applied successfully to solve this problem also.

10.1 The Green's function for the unbounded electro-elastic medium

Consider a uniform elastic piezoelectric material under isothermal conditions. The linear constitutive relations of electro-elasticity for this material have the form

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} - e_{ijk}E_k$$

$$D_i = e^T_{ikl}\varepsilon_{kl} + \eta_{ik}E_k \tag{10.1}$$

where σ and ε are the stress and strain tensors, E and D the electric field and electric displacement vectors, C is the tensor of elastic moduli at fixed electric field, η is the dielectric tensor (permittivity) at fixed strain, e is the tensor of piezoelectric constants characterizing coupled electro-elastic effects (superscript "T " denotes the transpose operation).
The fields σ and D satisfy the equations of the elastic and electric equilibrium

$$\nabla_j\sigma_{ij} = -Q_i, \quad \nabla_j D_j = -q \tag{10.2}$$

where Q is the density of the body forces and q is the density of the electric charges distributed in some closed domain V and supposed to be equal to zero outside of it. With the help of relations

$$\varepsilon_{ij} = \nabla_{(i}u_{j)}, \quad E_i = -\nabla_i\varphi \tag{10.3}$$

where u is the elastic displacement and φ is the electric potential and Eqs. (10.1) we can rewrite Eqs (10.2) in the form

$$\nabla_j C_{ijkl}\nabla_k u_l + \nabla_j e_{ijk}\nabla_k\varphi = -Q_i$$

$$\nabla_i e^T_{ikl} \nabla_k u_l - \nabla_i \eta_{ik} \nabla_k \varphi = -q \tag{10.4}$$

The solution vanishing at infinity of the system (10.4) can be presented as

$$u_i(x) = \int_V G_{ik}(x - x') Q_k(x') dx' + \int_V \Gamma_i(x - x') q(x') dx'$$

$$\varphi(x) = \int_V \gamma_k(x - x') Q_k(x') dx' + \int_V g(x - x') q(x') dx'. \tag{10.5}$$

Substitution of these expressions into (10.4) leads to the system of differential equations for the kernels $G_{ik}(x), \Gamma_i(x), \gamma_k(x)$ and $g(x)$

$$\nabla_j C_{ijkl} \nabla_k G_{lm}(x) + \nabla_j e_{ijk} \nabla_k \gamma_m(x) = -\delta_{im}\delta(x)$$

$$\nabla_j C_{ijkl} \nabla_k \Gamma_l(x) + \nabla_j e_{ijk} \nabla_k g(x) = 0$$

$$\nabla_i e^T_{ikl} \nabla_k G_{lm}(x) - \nabla_i \eta_{ik} \nabla_k \gamma_m(x) = 0$$

$$\nabla_i e^T_{ikl} \nabla_k \Gamma_l(x) - \nabla_i \eta_{ik} \nabla_k g(x) = -\delta(x) \tag{10.6}$$

The Fourier transformation of these equations gives

$$\Lambda_{il}(k) G_{lj}(k) + h_i(k) \gamma_j(k) = \delta_{ij}$$

$$h^T_l(k) G_{lj}(k) - \lambda(k) \gamma_j(k) = 0$$

$$\Lambda_{il}(k) \Gamma_l(k) + h_i(k) g(k) = 0$$

$$h^T_l(k) \Gamma_l(k) - \lambda(k) g(k) = 1 \tag{10.7}$$

where

$$\Lambda_{il} = k_j C_{ijkl} k_k, \quad h_i(k) = e_{ikl} k_k k_l, \quad h^T_l(k) = e^T_{ikl} k_i k_k, \quad \lambda(k) = \eta_{ik} k_i k_k. \tag{10.8}$$

The formal solution of this system of equations can be represented in the form

$$\mathcal{G}(k) = \left\| \begin{array}{cc} G_{ik}(k) & \Gamma_i(k) \\ \gamma_k(k) & -g(k) \end{array} \right\| \tag{10.9}$$

$$G_{ik} = \left(\Lambda_{ik} + \frac{1}{\lambda} h_i h_k^T\right)^{-1}, \quad g = (\lambda + h_i^T \Lambda_{ij}^{-1} h_j)^{-1}$$

$$\gamma_i = \frac{1}{\lambda} h_k^T G_{ki}, \quad \Gamma_i = \Lambda_{ik}^{-1} h_k g \tag{10.10}$$

It is not difficult to show that $\gamma_i = \Gamma_i$ hence, it can be written finally as:

$$\mathcal{G}(k) = \left\| \begin{array}{cc} G_{ik}(k) & \gamma_i(k) \\ \gamma_k^T(k) & -g(k) \end{array} \right\| \tag{10.11}$$

The x- presentation of the Green's function can be obtained with the help of inverse Fourier transformation

$$\mathcal{G}(x) = \frac{1}{(2\pi)^3} \int \mathcal{G}(k) e^{-i\mathbf{k}\cdot\mathbf{x}} d\mathbf{k} \tag{10.12}$$

Because the function $\mathcal{G}(k)$ permits the representation

$$\mathcal{G}(k) = k^2 \mathcal{G}(\xi), \quad \xi = k/|k|$$

it can be shown the same way as above (see Sec. 9.1) that Eq. (10.12) is transformed to

$$\mathcal{G}(x) = \frac{1}{8\pi^2} \int\limits_{S_1} \mathcal{G}(\xi) \delta(\xi \cdot x) dS_\xi \tag{10.13}$$

10.2 The solution of the electro-elastic problem for a single inclusion

The relations (10.1) can be conveniently written in the following short form

$$J = \mathbf{L}F, \quad J = \left\| \begin{array}{c} \sigma \\ D \end{array} \right\|, \quad \mathbf{L} = \left\| \begin{array}{cc} C & e \\ e^T & -\eta \end{array} \right\|, \quad F = \left\| \begin{array}{c} \varepsilon \\ -E \end{array} \right\| \tag{10.14}$$

where the symbolic "matrix" $\mathbf{L}$ must be regarded as linear operator, which converts the tensor-vector pair $[\sigma, D]$ into the analogous pair $[\varepsilon, F]$.

The relations inverse to (10.14) and giving F as a function of J are

$$F = \mathbf{M}J, \quad \mathbf{M} = \left\| \begin{array}{cc} S & d \\ d^T & -\kappa \end{array} \right\| \tag{10.15}$$

where

$$S = (C + e\eta^{-1}e^T)^{-1}, \quad \kappa = (\eta + e^T C^{-1} e)^{-1}, \quad d = Se\eta^{-1} = C^{-1}e\kappa$$

Let us consider now an infinite homogeneous piezoelectric body with operator of electro-elastic characteristics $\mathbf{L}^0$, containing the inclusion occupying a region V with electro-elastic constants $\mathbf{L}$. We start with the following system of elastic and electric equilibrium equations of the coupled electro-elastic theory for a heterogeneous medium

$$\nabla \mathbf{L}(x) \nabla f(x) = 0, \quad f(x) = \left\| \begin{array}{c} u_i(x) \\ \varphi(x) \end{array} \right\| \tag{10.16}$$

We can write the operator $\mathbf{L}(x)$ in the form

$$\mathbf{L}(x) = \mathbf{L}^0 + \mathbf{L}^1 V(x), \quad \mathbf{L}^1 = \mathbf{L} - \mathbf{L}^0 \tag{10.16}$$

where $V(x)$ is characteristic function of the region V. This representation allows us to reduce the system of differential equations (10.16) to a system of integral equation

$$F(x) = F^0(x) + \int_V \mathbf{P}(x - x')\mathbf{L}^1 F(x')dx' \tag{10.17}$$

$$\mathbf{P}(x) = \mathbf{D}\mathcal{G}(x)\mathbf{D}, \quad \mathbf{D} = \left\| \begin{array}{cc} def & 0 \\ 0 & -grad \end{array} \right\|.$$

Here $F^0(x)$ is the vector of external elastic and electric fields, which would be present in the medium without the inclusion and def is the symmetric part of the gradient, $\frac{1}{2}(\nabla_i + \nabla_j)$.

For $x \in V$ the system of equations (10.17) yields the fields $\varepsilon(x)$ and $E(x)$ inside the inclusion, thereupon the fields outside of V are determined uniquely.

Let now the inclusion have an ellipsoidal shape with semi-axes a_1, a_2, a_3. Then the domain V is defined by the relations: $x_i(a^{-2})_{ij}x_j \leq 1, \quad a_{ij} = a_i\delta_{ij}$ (no summing with respect to i!). In this case it is possible to prove the analogue of Eshelby's theorem considered above: *If the external electro-elastic fields are uniform in the domain V ($F^0 = const$), the electro-elastic fields F inside V are also uniform.* It can be proved in the same way as above but we present here another proof based on so-called Radon transformation.

Let the domain V be a sphere with radius a centered at the origin. If $F = const.$ in V, the problem reduces to the evaluation of the integral

$$\int_V \mathbf{P}(x - x')dx' = \nabla \otimes \nabla \frac{1}{8\pi^2} \int_V dx \int_{S_1} \mathcal{G}(\xi)\delta(x \cdot \xi) dS_\xi =$$

$$= \frac{1}{8\pi^2} \int_{|\xi|=1} \mathbf{P}(\xi) dS_\xi \cdot \frac{\partial^2}{\partial p^2} \int_V \delta(p - \xi \cdot x')dx' \qquad (10.18)$$

$$p = \xi \cdot x, \quad \mathbf{P}(\xi) = \xi \mathcal{G}(\xi)\xi.$$

The integral over the domain V in (10.18) is equal to the area of intersection of the plane $\xi \cdot x = p$ with the sphere V; that is $\pi(a^2 - p^2)$ if $|p| \leq a$ and zero if $p > a$. For $x \in V$, the second derivative of this integral equals -2π and the right-hand side of (10.18) is a constant. Similar results can be obtained for ellipsoids, which by means of a coordinate transformation $y_i = a_{ij}^{-1} x_j$ can be transformed to the unit sphere. In this case

$$\int_V \mathbf{P}(x - x')dx' = -\mathbf{P}^0 = const \qquad (10.19)$$

$$\mathbf{P}^0 = -\frac{|\det a|}{4\pi} \int_{|\xi|=1} \mathbf{P}(\xi) \frac{dS_\xi}{\rho^3(\xi)}, \quad \rho(\xi) = \sqrt{\xi_i (a^2)_{ij} \xi_j}.$$

Thus, for the external field F^0 constant in V the integral equation (10.17) is transformed into the algebraic equation

$$F = F^0 - \mathbf{P}^0 \mathbf{L}^1 F \qquad (10.20)$$

Solving this equation for F, we can express the strain field ε and the electric field E via external fields ε^0 and E^0

$$F = \mathbf{A}F^0, \quad \mathbf{A} = (\mathbf{I} + \mathbf{P}^0 \mathbf{L}^1)^{-1}, \quad \mathbf{I} = \left\| \begin{array}{cc} I_{ijkl} & 0 \\ 0 & \delta_{ik} \end{array} \right\| \qquad (10.21)$$

10.3 Overall electro-elastic properties of a composite material

We examine now an unbounded piezoelectric medium with properties $\mathbf{L}^0$, containing a spatially homogeneous random set of ellipsoidal inclusions which occupy a system of isolated regions V_k with characteristic functions $V_k(x), k = 1, 2, \dots$. The system of equations for the strain field $\varepsilon(x)$ and the electric field $E(x)$ in the medium with inhomogeneities can be written in a form similar to (10.17)

$$F(x) = F^0(x) + \int \mathbf{P}(x - x')\mathbf{L}^1(x')F(x')V(x')dx' \tag{10.22}$$

where $V(x)$ denotes the characteristic function of the region $V = \sum_k V_k$, occupied by all inclusions.

It is not difficult to see that from the formal point of view the determination of electro-elastic fields ε and E is quite similar to the uncoupled elastic problem considered above . Therefore, dropping the analogous details including the introduction of the local effective field $F^*(x)$ and average procedure we present here only the final result. The average electro-elastic fields $\langle f(x)\rangle$ in the composite satisfy the equation

$$\nabla \mathbf{L}^* \nabla \langle f(x)\rangle = 0 \tag{10.23}$$

$$\mathbf{L}^* = \mathbf{L}^0 + \mathbf{D}\overline{\mathbf{L}}^A, \quad \overline{\mathbf{L}}^A = \left\langle \mathbf{L}^A(x)V(x)\right\rangle, \quad \mathbf{L}^A(x) = \mathbf{L}^1(x)\mathbf{A}(x)$$

$$\mathbf{D} = (\mathbf{I} - \left\langle \mathbf{L}^A(x)V(x)\mathbf{P}^\Phi_a\right\rangle)^{-1}, \quad \mathbf{P}^\Phi_a = -\int P(x)\Phi_a(x)dx$$

where the operator $\mathbf{A}$ is determined in (10.21) and it is denoted : $\Phi_a(x) = 1 - \langle V(x)\rangle^{-1} \langle V(x;x')|x,a\rangle$.

The quantity $\mathbf{L}^*$ represents the operator of effective electro-elastic characteristics of piezoelectric composites.

Let us consider a special case.

10.3.1 Transversely isotropic piezoelectric material reinforced by the continuous fibers

We consider a composite material whose matrix is transversely isotropic. The tensors C^0, e^0 and η^0 for such a medium can be represented in the form

$$C^0 = k_0T^2 + 2m_0\left(T^1 - \frac{1}{2}T^2\right) + l_0(T^3 + T^4) + 4\mu_0T^5 + n_0T^6$$

$$e = e_1^0 U^1 + e_2^0 U^2 + e_3^0 U^3, \quad \eta = \eta_1^0 t^1 + \eta_2^0 t^3 \tag{10.24}$$

Here $k_0, m_0, l_0, \mu_0, n_0$ are the five independent elastic moduli of the transversely isotropic medium, e_1^0, e_2^0, e_3^0 are three piezoelastic constants and η_1^0, η_2^0 are two permittivities. The quantities T^i were introduced above (see (2.70)) , U^k, t^m are the elements of the tensor bases, given by the expressions

$$U_{ijk}^1 = \theta_{ij} m_k, \quad U_{ijk}^2 = 2m_{(i}\theta_{j)k}, \quad U_{ijk}^3 = m_i m_j m_k$$

$$t_{ij}^1 = m_i m_j, \quad t_{ij}^2 = \theta_{ij}, \quad \theta_{ij} = \delta_{ij} - m_i m_j \tag{10.25}$$

where m is the unit vector of the axis of symmetry of the material.
Suppose the inclusions in the composite have the form of continuous cylinders of the same radius similarly oriented parallel to the axis of symmetry of the properties of the matrix (a medium reinforced by unidirectional continuous fibers). To determine the operator $\mathbf{P}^0$ in this case we will use the general expression (10.19) and assume the inclusion is a prolate spheroid ($a_1 = a_2 = a$, $a_3 > a$). We will change in (10.19) to a spherical system of coordinates φ, θ with polar axis directed along the axis of the spheroid. We make the change $\cos\theta = t$. We can then write

$$\mathbf{P}^0 = -\frac{1}{2\pi}\int_0^{2\pi} d\varphi \int_{-1}^{1} \mathbf{P}(\varphi, t)\psi(t,\delta)dt \tag{10.25}$$

$$\psi(t,\delta) = \frac{1}{2}\delta^2[\delta^2 + (1-\delta^2)t^2]^{-\frac{3}{2}}, \quad \delta = \frac{a}{a_3}$$

If we take the limit as the aspect ratio $\delta \to 0$ this corresponds to an inclusion in the form of an infinite circular cylinder (a fiber) of radius a. Taking this limit in (10.25) and taking into account that

$$\lim_{\delta\to 0} \frac{\delta^2}{2[\delta^2 + (1-\delta^2)t^2]^{\frac{3}{2}}} = \delta(t)$$

we obtain

$$\mathbf{P}^0 = -\frac{1}{2\pi}\int_0^{2\pi} \mathbf{P}(\varphi, 0)d\varphi \tag{10.26}$$

To calculate the tensors occurring in this operator in explicit form we need to put $\xi_3 = 0$ in the tensors of $\mathbf{P}$ in (10.19) (the vector ξ is then in the plane perpendicular to the fiber axis), substitute the expression obtained into (10.26) and evaluate the integrals.

For a transversely isotropic medium the components of the matrix $\mathcal{G}(\xi)$ in (10.13) when $\xi = (\xi_1, \xi_2, 0)$, have the form

$$G_{ij}(\xi) = -\frac{k_0}{m_0(k_0+m_0)}\xi_i\xi_j + \frac{1}{m_0}\theta_{ij} + \frac{1}{\mu_0'}m_i m_j \tag{10.27}$$

$$\gamma_i = \gamma m_i, \quad g = -\frac{1}{\eta_2'}, \quad \gamma = \frac{e_2^0}{\mu_0\eta_2^0 + (e_2^0)^2}$$

$$\mu_0' = \mu_0 + \frac{(e_2^0)^2}{\eta_2^0}, \quad \eta_2' = \eta_2^0 + \frac{(e_2^0)^2}{\mu_0}$$

Substituting (10.27) into (10.26) and integrating over the unit circle, we obtain

$$\mathbf{P}^0 = \left\| \begin{array}{cc} P & r \\ -r^T & p \end{array} \right\| \tag{10.28}$$

$$P = P_1T^2 + P_2\left(T^1 - \frac{1}{2}T^2\right) + \frac{1}{2\mu_0'}T^5, \quad r = \frac{1}{4}\gamma U^2$$

$$p = \frac{1}{2\eta_2'}t^2, \quad P_1 = \frac{1}{4(k_0+m_0)}, \quad P_2 = \frac{k_0+2m_0}{4m_0(k_0+m_0)}$$

If we assume that the correlation hole also has the form of a cylinder, parallel to the fibers, the general formula (10.23) will take the form

$$\mathbf{L}^* = \mathbf{L}^0 + p\mathbf{L}^1[\mathbf{I} + (1-p)\mathbf{P}\mathbf{L}^1]^{-1} \tag{10.29}$$

Suppose the fiber is also transversely isotropic with the axis of symmetry of the properties coinciding with their geometrical axis. The tensors of the electro-elastic characteristics for this material are defined by the same formulae (10.24), in which we must omit the zero subscript on the physical constants.

To perform the tensor operations in formula (10.29) it is very convenient to use the relations (2.73) and (2.74). Besides them, we present here some of other useful relations for operations with tensors presented in T, U and t-basis.

Let two tensors C and D be represented in U-basis as:

$$C_{ijk} = \sum_{r=1}^{3} C_r U_{ijk}^r, \quad D_{ijk} = \sum_{s=1}^{3} D_s U_{ijk}^s$$

The product-contraction of these tensors over one index gives the tensor in T-basis

$$C_{ijm}D^T_{mkl} = C_1D_1T^2_{ijkl} + C_1D_3T^3_{ijkl} + C_3D_1T^4_{ijkl} + 4C_2D_2T^5_{ijkl} + C_3D_3T^6_{ijkl} \tag{10.30}$$

The contraction of the tensors C and D over two indices gives the tensor represented in t-basis as

$$C^T_{ikl}D_{klj} = 2C_2D_2t^1_{ij} + (2C_1D_1 + C_3D_3)t^2_{ij} \tag{10.31}$$

The t-basis is orthogonal, i.e. if

$$\alpha_{ij} = \alpha_1 t^1_{ij} + \alpha_2 t^2_{ij}, \quad \beta_{ij} = \beta_1 t^1_{ij} + \beta_2 t^2_{ij}$$

then

$$\alpha_{ik}\beta_{kj} = \alpha_1\beta_1 t^1_{ij} + \alpha_2\beta_2 t^2_{ij} \tag{10.32}$$

and

$$\alpha^{-1}_{ij} = \frac{1}{\alpha_1}t^1_{ij} + \frac{1}{\alpha_2}t^2_{ij} \tag{10.33}$$

It remains only to present the results for the mixed contraction of the tensors presented in T, U and t-basis

$$A_{ijmn}C_{mnk} = (2A_1C_1 + A_3C_3)U^1_{ijk} + \frac{1}{2}A_5C_2U^2_{ijk} + (2A_4C_1 + A_6C_3)U^3_{ijk}$$

$$C^T_{imn}A_{mnkl} = (2C_1A_1 + C_3A_4)U^{1T}_{ijk} + \frac{1}{2}C_2A_5U^{2T}_{ijk} + (2C_1A_3 + C_3A_6)U^{3T}_{ijk}$$

$$\alpha_{im}C^T_{mkl} = \alpha_2C_1U^{1T}_{ijk} + \alpha_1C_2U^{2T}_{ijk} + \alpha_2C_3U^{3T}_{ijk} \tag{10.34}$$

$$C_{ijm}\alpha_{mk} = C_1\alpha_2U^1_{ijk} + C_2\alpha_1U^2_{ijk} + C_3\alpha_2U^3_{ijk}$$

With the help of these formulae we obtain from (10.29) that the composite as a whole will be transversely isotropic and will be characterized by the following five effective elastic moduli:

$$k^* = k_0 + pk_1d(p), \quad m^* = m_0 + pm_1\left[1 + (1-p)\frac{m_1(k_0 + 2m_0)}{2m_0(k_0 + m_0)}\right]^{-1}$$

$$l^* = l_0 + pl_1 d(p), \quad \mu^* = \mu_0 + \frac{p}{\Delta(p)}\left[\mu_1 + \frac{(1-p)f}{2\eta_2'}\right]$$

$$n^* = n_0 + p\left[n_1 - \frac{(1-p)l_1^2 d(p)}{k_0 + m_0}\right], \quad d(p) = \frac{k_0 + m_0}{k_0 + m_0 + (1-p)k_1} \tag{10.35}$$

$$\Delta(p) = [1 + (1-p)b][1 + (1-p)B] - (1-p)^2 Qq, \quad f = \mu_1\eta_2^1 + (e_2^1)^2$$

$$b = \frac{1}{2}\left(\frac{\eta_2^1}{\eta_2'} + \gamma e_2^1\right), \quad B = \frac{1}{2}\left(\frac{\mu_1}{\mu_0'} + \gamma e_2^1\right),$$

$$q = \frac{1}{2}\left(\frac{e_2^1}{\eta_2'} - \gamma\mu_1\right), \quad Q = \frac{1}{2}\left(\gamma\eta_2^1 - \frac{e_2^1}{\mu_0'}\right)$$

three piezoelastic constants:

$$e_1^* = e_1^0 + pe_1^1 d(p), \quad e_2^* = e_2^0 + \frac{p}{\Delta(p)}\left[e_2^1 + \frac{1}{2}(1-p)\gamma f\right]$$

$$e_3^* = e_3^0 + p\left[e_3^1 - \frac{(1-p)l_1 e_1^1 d(p)}{k_0 + m_0}\right] \tag{10.36}$$

and two permittivities:

$$\eta_1^* = \eta_1^0 + p\left[\eta_1^1 + \frac{(1-p)(e_1^1)^2 d(p)}{k_0 + m_0}\right]$$

$$\eta_2^* = \eta_2^0 + \frac{p}{\Delta(p)}\left[\eta_2^1 + \frac{(1-p)f}{2\mu_0'}\right] \tag{10.37}$$

It can be seen from these formulae that taking into account the coupling of the elastic and electric fields affects only the values of the effective elastic modulus μ^*, the piezoelastic constants and the permittivities. As regards the elastic moduli of the composite k^*, m^*, l^*, n^*, they are the same as obtained earlier for the uncoupled elasticity.

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