MELNIKOV METHODS AND CHAOS IN DIFFERENTIAL EQUATIONS

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PREFACE

In May, 1998 I was invited by the Department of Mathematics and Mechanics within IIMAS at the National University of Mexico in Mexico City to give a series of four lectures on Melnikov methods. My assignment was to give one lecture accessible to undergraduates, a two-lecture summary of existing Melnikov techniques, and one lecture on my current research. While it was not required that the lectures be connected, I was pleased to be able to prepare four talks which formed a coherent whole.

At the end of my visit I agreed to prepare written lecture notes sumarizing the talks and that is what appears here with each lecture labeled as a chapter.

CHAPTER I

THE SEARCH FOR CHAOS IN PARTIAL DIFFERENTIAL EQUATIONS BY BUBNOV-GALERKIN-KANTOROVICH REDUCTION

The Bubnov-Galerkin Method

The topic of chaos is a very popular and important subject in the areas of dynamical systems and differential equations. In Chapters II and III we give a precise definition of chaos and present techniques for proving the existence of chaos for ordinary differential equations. Unfortunately, many, if not most, interesting applications are described by partial differential equations and these involve another level of difficulty. Over the last eighty years various techniques have been developed for the reduction of partial differential equations to problems of algebra or to problems of comparatively simple ordinary differential equations. The purpose of this chapter is to present one of these techniques in a purely formal way.

The method we shall describe is usually referred to as the Galerkin method after the work published in 1915 by B. G. Galerkin [9] although the idea behind the method was used two years earlier by I. G. Bubnov [2]. A nice history of these ideas appears in the historical introduction to the book by S. G. Mikhlin [24] who uses the term "Bubnov-Galerkin method".

The procedure of the Bubnov-Galerkin method is to reduce a boundary value problem in partial differential equations to an algebra problem. The classic problems of decades past involved linear partial differential equations in which case the result of the Bubnov-Galerkin method is a system of linear algebraic equations.

To illustrate the Bubnov-Galerkin method it is sufficient to consider an ordinary differential equation. For example, consider the boundary value problem

$$y'' + 2y' + xy = x, \quad y(0) = 0,\, y(1) = 0. \tag{1.1}$$

We seek a function $y = f(x)$ satisfying the differential equation and the given boundary conditions. Let us look for a solution in the form of an infinite series $y = \sum_{n=1}^{\infty} c_n \sin n\pi x$. Notice the functions $\sin n\pi x$ satisfy the boundary conditions. It remains to determine the constants c_n so that the series satisfies the differential equation.

Substituting the series into the differential equation yields

$$\sum_{n=1}^{\infty} -n^2\pi^2 c_n \sin n\pi x + 2\sum_{n=0}^{\infty} n\pi c_n \cos n\pi x + \sum_{n=0}^{\infty} c_n x \sin n\pi x = x.$$

We now multiply the preceding equation by $\sin \pi x$ and integrate from 0 to 1 to get one algebraic equation. We then repeat the process with $\sin 2\pi x$ to get our second equation and so on. The result of this is an infinite set of algebraic equations for an infinite set of unknowns. The equations have the form

$$\begin{array}{ccccccc} \left(\frac{1-2\pi^2}{4}\right) c_1 & - & \left(\frac{8+24\pi^2}{9\pi^2}\right) c_2 & + & \dots & = & \frac{1}{\pi}, \\ \left(\frac{24\pi^2-8}{9\pi^2}\right) c_1 & + & \left(\frac{1-8\pi^2}{4}\right) c_2 & + & \dots & = & -\frac{1}{2\pi}, \\ \vdots & & \vdots & & & & \vdots \end{array}$$

We obtain an approximate solution to the problem by truncating the series to get a finite number of algebraic equations. If we use one term then we must solve only $\left(\frac{1-2\pi^2}{4}\right) c_1 = 1/\pi$. This gives $c_1 = -.0679452$ and the approximate solution $y = -.0679452 \sin x$.

If we keep two terms in the series we must solve

$$\begin{array}{ccccc} \left(\frac{1-2\pi^2}{4}\right) c_1 & - & \left(\frac{8+24\pi^2}{9\pi^2}\right) c_2 & = & \frac{1}{\pi}, \\ \left(\frac{24\pi^2-8}{9\pi^2}\right) c_1 & + & \left(\frac{1-8\pi^2}{4}\right) c_2 & = & -\frac{1}{2\pi}. \end{array}$$

The solution to these equations is $c_1 = -.0674994$, $c_2 = -.000757563$ and as an approximate solution to the boundary value problem

$$y = -.0674994 \sin x - .000757563 \sin 2x$$

This can be extended to any degree of accuracy.

The Bubnov-Galerkin method can be used for partial differential equations provided one has a complete set of functions which satisfy a set of homogeneous boundary conditions. For example, suppose one seeks a function $w = f(x, y)$ satisfying $w(0, y) = w(1, y) = w(x, 0) = w(x, 1) = 0$. In other words, w is required to be zero on the boundary of the unit square. One could then write

$$w(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c_{mn} \sin mx \sin ny.$$

One then substitutes this into the partial differential equation, multiplies by $\sin ix \sin jy$ for arbitrary i, j and integrates over the unit square to get a system of algebraic equations for the coefficients c_{mn}.

The Bubnov-Galerkin-Kantorovich Method

In 1933, L. V. Kantorovich [19] proposed a modification of the Bubnov-Galerkin method. In the book by Kantorovich and Krylov [20] one can find a description and example of this method which they refer to as Kantorovich's method. The name Bubnov-Galerkin-Kantorovich seems more appropriate.

The idea of the Bubnov-Galerkin-Kantorovich method is to assume a solution in which the unknown coefficients are, not constants, but rather functions of one of the variables. After integrating out the remaining variables one is left with a system of ordinary differential equations for the coefficient functions.

As an example consider the problem

$$x\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = \sin t, \tag{1.2}$$
$$u(0,t) = u(\pi,t) = 0,\ u(x,0) = u_0(x),\ \frac{\partial u}{\partial t}(x,0) = v_0(t).$$

This can be viewed as the problem of a vibrating string with variable density $\rho(x) = x$, fixed at the ends $x = 0$ and $x = \pi$, with initial displacement $u_0(x)$ and initial velocity $v_0(x)$.

We assume a solution to this problem of the form

$$u(x,t) = \sum_{n=1}^{\infty} w_n(t) \sin nx. \tag{1.3}$$

We see that this form satisfies the boundary conditions at $x = 0$ and $x = \pi$. The coefficient functions, w_n, must be determined so as to satisfy the differential equation and the initial conditions.

We substitute (1.3) into the differential equation to obtain

$$\sum_{n=1}^{\infty} \ddot{w}_n x \sin nx + \sum_{n-1}^{\infty} n^2 w_n \sin nx = \sin t$$

where $\ddot{w}_n = \frac{d^2 w_n}{dx^2}$.

In the preceding equation we multiply by $\sin x$ and integrate from $x = 0$ to $x = \pi$, multiply by $\sin 2x$ and integrate again, etc. The result is an infinite system of differential equations for the functions w_n. Here are the first three equations:

$$\begin{array}{rcrcrcccccl}
\frac{\pi^2}{4}\ddot{w}_1 & - & \frac{8}{9}\ddot{w}_2 & - & \frac{16}{225}\ddot{w}_4 & + & \dots & + & \frac{\pi}{2}w_1 & = & \sin t, \\
-\frac{8}{9}\ddot{w}_1 & + & \frac{\pi^2}{4}\ddot{w}_2 & - & \frac{24}{25}\ddot{w}_3 & + & \dots & + & 2\pi w_2 & = & 0, \\
 & & -\frac{24}{25}\ddot{w}_2 & + & \frac{\pi^2}{4}\ddot{w}_3 & + & \dots & + & \frac{9\pi}{2}w_3 & = & \frac{2}{3}\sin t.
\end{array}$$

As before, we obtain an approximate solution by truncating the series and using a corresponding number of equations. If we keep only the first term we need only solve

$$\frac{\pi^2}{4}\ddot{w}_1 + \frac{\pi}{2}w_1 = \sin t.$$

This is a simple forced harmonic oscillator. After solving this subject to the initial equations the approximate solution to (1.2) is $w(x,t) = w_1(t)\sin x$.

To get a more accurate solution we might keep two terms in which case we would have to solve

$$\begin{array}{rcrcrcl}
\frac{\pi^2}{4}\ddot{w}_1 & - & \frac{8}{9}\ddot{w}_2 & + & \frac{\pi}{2}w_1 & = & \sin t, \\
-\frac{8}{9}\ddot{w}_1 & + & \frac{\pi^2}{4}\ddot{w}_2 & + & 2\pi w_2 & = & 0.
\end{array}$$

Here we have a system of ordinary differential equations to solve. However, these are linear with constant coefficients and so their solution is an exercise in linear algebra discussed in many undergraduate textbooks. After solving the preceding equation the approximate solution to (1.2) is $w(x,t) = w_1(t)\sin x + w_2(t)\sin 2x$.

The examples we have discussed so far involved linear partial differential equations. For this reason, the reduction technique yielded linear algebraic equation (Bubnov-Galerkin) or linear ordinary differential equation (Bubnov-Galerkin-Kantorovich). Recent research problems involve nonlinear partial differential equations and a consequent reduction to nonlinear ordinary differential equations. In this chapter we give a formal description of this process for some classic problems. In Chapter IV we give a more rigorous discussion of some ongoing research using this technique.

The Lorenz Equation

Possibly the best known example of chaos in nonlinear partial differential are the so-called Lorenz equations named for the work of E. N. Lorenz [22]. These equations arise from a model for convection in a viscous fluid. Our description here is taken from the book by J. K. Bhattacharjee [1] where further details can be found.

When a layer of fluid is heated on its lower surface the warmer fluid will rise and conduct heat upward. It has been observed that this occurs in what are called convection rolls as illustrated in Fig. 1.1.

FIGURE 1.1. Convection Rolls for Lorenz Model

The governing equations are the Navier-Stokes equations and the heat conduction equation which are, respectively,

(1.4a)
$$\nabla^2 \left(\frac{\partial}{\partial t} - \nabla^2\right) w = R \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \theta + \frac{1}{\sigma} \left(\nabla \times \nabla \times (\Omega \times \mathbf{u})\right)_z ,$$

(1.4b)
$$\left(\sigma \frac{\partial}{\partial t} - \nabla^2\right) \theta = w - (\mathbf{u} \cdot \nabla)\theta.$$

The terminology here is:

$\mathbf{u} = (u, v, w)$ the velocity vector in the fluid,

$\theta =$ temperature,
$R =$ Rayleigh number,
$\sigma =$ viscosity/diffusivity.

For simplicity we assume that the flow is planer and uniform in y so that $v = 0$ and u, v, and θ are functions of (x, z, t). We now assume a solution of the form

$$\begin{aligned} u(x,z,t) &= a(t)\sin a_0 x \cos \pi z, \\ w(x,z,t) &= -\frac{a_0}{\pi} a(t) \cos a_0 x \sin \pi z, \\ \theta(x,z,t) &= b(t)\cos \alpha_0 x \sin \pi z + c(t) \sin 2\pi z \end{aligned}$$

where $a_0^2 = \pi^2/2$ and a, b, c are undetermined functions of t. For details see [1].

The preceding formulas are substituted into (1.4). To obtain three ordinary differential equations we next multiply (1.4a) by $\cos a_0 x \sin \pi z$, multiply (1.4b) by $\cos a_0 x \sin \pi z$ and also multiply (1.5b) by $\sin 2\pi z$. Each of the resulting equations is then integrated from $-\pi/a_0$ to π/a_0 for x and from 0 to 1 for z. The result is

$$\begin{aligned} \dot{a} &= -(\pi^2 + a_0^2)a + \frac{R a_0^2 \pi}{\pi^2 + a_0^2} b, \\ \sigma \dot{b} &= -(\pi^2 + a_0^2) b - \frac{a}{\pi} - ac, \\ \sigma \dot{c} &= -4\pi^2 c + \frac{ab}{2}. \end{aligned}$$

Finally, we rescale a, b, c and t and name the rescaled dependant variables x, y, z. The result is

$$\begin{aligned} \dot{x} &= \sigma(-x + y), \\ \dot{y} &= -xz + rx - y, \\ \dot{z} &= xy - bz \end{aligned}$$

where $b = 8/3$ and $r = (4R)/(27\pi^4)$.

These are the famous Lorenz equations [22].

An Oscillating Beam

As another example consider an elastic beam along the z axis compressed by an axial load Γ and with pinned ends at $z = 0$ and $z = 1$. If the beam undergoes transverse oscillation with the deflection denoted $u(z, t)$ then the governing differential equation of motion is

$$\begin{gathered} \ddot{u} = u'''' + \left[\Gamma - \int_0^1 \left[u'(z)\right]^2 \, dz\right] u'' + \mu_2 \dot{x} = \mu_1 \cos \omega t, \\ u(0,t) = u''(0,t) = u(1,t) = u''(1,t) = 0. \end{gathered}$$

Here, a dot denotes differentiation with respect to t, prime with respect to z. The $\ddot{u}$ and u'''' terms represent inertia and bending stiffness, respectively and the μ_1, μ_2 terms damping and an applied transverse forcing. The Γ term is the transverse shear due to the axial load while the integral term arises from the reduction in axial load due to elongation of the beam due to bending. The boundary conditions are the standard ones for a beam with pinned ends.

This problem was considered by Holmes [17] and since by many others. His approach was to make a Bubnov-Galerkin-Kantorovich expansion of the form $u(z,t) = \sum_{k=1}^{\infty} x_k(t) \sin k\pi z$. We substitute this into the differential equation, multiply by $\sin n\pi z$ for arbitrary n and integrate from $z = 0$ to $z = 1$. It is surprising easy to verify the resulting set of equations:

$$\ddot{x}_n + (n\pi)^2 \left[(n\pi)^2 - \Gamma\right] x_n + \frac{\pi^2}{2}(n\pi)^2 x_n \sum_{n=1}^{\infty} n^2 x_n^2 + \mu_2 \dot{x}_n = 2\mu_1 \left[\frac{1-(-1)^n}{n\pi}\right] \cos \omega t$$

with one equation for each $n = 1, 2, \ldots$.

We shall consider the μ_i terms as perturbation terms. The linear part of the unperturbed nth equation is $\ddot{x}_n + (n\pi)^2 \left[(k\pi)^2 - \Gamma\right] x_n = 0$. The equations naturally divide into two types according to whether $(n\pi)^2 - \Gamma < 0$ or $(n\pi)^2 - \Gamma \geq 0$. The former represents a hyperbolic equilibrium while the latter is a center. The number of equations with a hyperbolic equilibrium is determined by the magnitude of Γ, the axial load. As Γ is increased from zero the first hyperbolic mode occurs at $\Gamma = \pi^2$, the Euler buckling load. Holmes assumes $\pi^2 < \Gamma < 4\pi^2$ so that there is one hyperbolic equation. He then truncates the series at one term to get rid of the center part. The resulting differential equation is

$$\ddot{x}_1 = \pi^2(\Gamma - \pi^2)x_1 - \frac{\pi^4}{2}x_1^3 - \mu_2 \dot{x}_1 + \frac{4}{\pi}\mu_1 \cos \omega t.$$

By rescaling x, t and the μ_is this equation can be put in the form

$$\ddot{x} = x - 2x^3 - \mu_2 \dot{x} + \mu_1 \cos \omega t.$$

This is the very well-known forced, damped Duffing equation with negative stiffness. This equation is discussed in many textbooks where it is shown using a technique often called Melnikov theory that the dynamics exhibit chaos. We shall not go into detail at this time but merely say that the basic idea is that when $\mu_1 = \mu_2 = 0$ the equation is autonomous with a homoclinic solution and for a range of nonzero parameter values the equation is nonautonomous with a transverse homoclinic orbit. The latter is a well known sufficient condition for chaos.

CHAPTER II

CHAOS IN ORDINARY DIFFERENTIAL EQUATIONS

Symbolic Dynamics

To give a rigorous treatment of chaos we first give a definition and then an existence proof. The beginning of the modern subject is usually taken to be the well known paper of Smale [27] where he developed the ideas of symbolic dynamics and applied them to dynamical systems. Since then these concepts have appeared in textbooks such as those by Z. Nitecki [25] and S. Wiggins [29] where details can be found. We shall give only a brief outline.

We first choose a positive integer N, define the set $S_N = \{0, 1, 2, \ldots, N-1\}$ and endow this set with the discrete topology. Next we define the product space $\Sigma_N = \Pi_{k=-\infty}^{\infty} X_k$ where each $X_k = S_N$. Each element $\sigma \in \Sigma_N$ has the form $\sigma = (\ldots, \sigma_{-2}, \sigma_{-1}, \sigma_0, \sigma_1, \sigma_2, \ldots)$ where each $\sigma_i \in S_N$. In other words, each $\sigma \in \Sigma_N$ is a doubly infinite sequence of integers between 0 and $N-1$. Actually, it is sufficient to take $N = 2$ and use doubly infinite sequences of zeros and ones but many authors prefer more generalization.

The space Σ_N has the product topology. It can be shown that this is also a metric space with metric

$$d(\sigma, \sigma') = \sum_{k=-\infty}^{\infty} \frac{\delta_k}{2|k|}, \quad \delta_k = \begin{cases} 0, & \text{if } \sigma_k = \sigma'_k \\ 1, & \text{if } \sigma_k \neq \sigma'_k. \end{cases}$$

Using these two characterizations of the topology one can show that Σ_N is compact, totally disconnected and perfect which is the definition of an abstract Cantor set. For definitions of these concepts see the textbook by Dugundji [7].

Define the map $\varphi : \Sigma_N \to \Sigma_N$ by $\varphi(\sigma)_i = \sigma_{i+1}$ which simply translates each sequence in Σ_N by one place. The function φ is called the Bernoulli shift on Σ_N. It is reasonably easy to show that φ has the following properties:

- φ is continuous.
- There exists a countable infinity of periodic orbits.
- There exists an uncountable infinity of nonperiodic orbits.
- There exists a dense orbit.

This last property shows that φ exhibits what is called sensitive dependence on initial condition. Let σ, σ' be any two points in Σ_N and let U, U' be arbitrary neighborhoods of σ, σ' respectively. Then the dense orbit carries a point in U to some point in U'. In other words, given $\sigma \in \Sigma_N$ there exists a point arbitrarily close to σ which travels arbitrarily close to any other point in Σ_N under iterates of φ

Quite irregular dynamics on the abstract space Σ_N arise when φ is iterated. The problem which we must address is showing that this chaotic structure has something to do with differential equations.

Exponential Dichotomies

As discussed in the next section, there are different techniques for relating shift dynamics to the flow of a differential equation. We shall use the machinery of exponential dichotomies which we describe here. The basic source for this material is the set of the lecture notes by W. A. Coppel [6].

Consider a linear differential equation

$$\dot{x} = A(t)x \tag{2.1}$$

where A is a continuous, $n \times n$ matrix-valued function of a real variable t.

2.1 DEFINITION. *We say that the triple (U, P_1, P_2) is an exponential dichotomy with constants (K, a) on the interval $[t_1, t_2]$ for (2.1) if K, a are two positive constants and the following hold:*

(i) *U is a fundamental solution for (2.1).*
(ii) *P_1, P_2 are projections on $\mathbb{R}^n$ with $P_1 + P_2 = I$.*
(iii) *The following inequalities hold:*

$$\left|U(t)P_1U(s)^{-1}\right| \le Ke^{a(s-t)} \quad \text{for } t_1 \le s \le t \le t_2,$$
$$\left|U(t)P_2U(s)^{-1}\right| \le Ke^{a(t-s)} \quad \text{for } t_1 \le t \le s \le t_2.$$

We allow $t_1 = -\infty$ and/or $t_2 = +\infty$ in which case the interval must be open at the corresponding end(s). If both of these hold we say (2.1) has an exponential dichotomy on the whole line. We often write (U, P) to mean $(U, P, I - P)$.

There is a standard example which clarifies these ideas. Consider the equation $\dot{x} = A_0x$ where A_0 is a constant, $n \times n$ matrix. Assume that A_0 is a hyperbolic matrix by which we mean that its eigenvalues lie off the imaginary axis. Then we can assume A_0 has the form $A_0 = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}$ where J_1, J_2 are square matrices whose eigenvalues satisfy $\Re(\lambda) \le -2a < 0$ and $\Re(\lambda) \ge 2a > 0$ respectively for some constant $a > 0$.

Define a fundamental solution, U, and projections P_1, P_2 as follows:

$$U(t) = e^{tA_0} = \begin{bmatrix} e^{tJ_1} & 0 \\ 0 & e^{tJ_2} \end{bmatrix}, \quad P_1 = \begin{bmatrix} I_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 & 0 \\ 0 & I_2 \end{bmatrix}$$

where I_k is the identity matrix with the same order as J_k. Note that we have

$$U(t)P_1U(s)^{-1} = \begin{bmatrix} e^{(t-s)J_1} & 0 \\ 0 & 0 \end{bmatrix}$$

from which we get $|U(t)P_1U(s)| \le Ke^{a(s-t)}$ for some $K > 0$ and all $s \le t$ and in a similar way we can satisfy the second required inequality. Thus, we have an exponential dichotomy on the whole line.

More generally, it is sufficient that the coefficient matrix in (2.1) approach a constant hyperbolic matrix as $t \to \infty$ as long as this occurs at an exponential rate. We state this precisely in the next theorem which is contained within §3.8 of Coddington and Levinson [5].

2.2 THEOREM. *If there exists a constant hyperbolic matrix* A_0 *and* $b > 0$ *such that* $\sup_{t\geq 0} |A(t) - A_0|e^{bt} < \infty$ *then (2.1) has an exponential dichotomy on* $[0, \infty)$.

The proof of this theorem is a nice exercise for the reader. The steps are: write the equation in the form $\dot{x} = A_0 x + B(t)$, choose $t_0 > 0$ so that $|B(t)|$ is small on $[t_0, \infty)$, use the exponential dichotomy for $\dot{x} = A_0 x$ and variation of constants to obtain an integral equation and solve with the contraction mapping theorem.

The principal use of an exponential dichotomy lies in solving certain linear non-homogeneous equations. Suppose (2.1) has an exponential dichotomy (U, P_1, P_2) with constants (K, a) on $[0, \infty)$, consider the equation

$$\dot{x} = A(t)x + w \tag{2.2}$$

and suppose $w \in \mathcal{C}^0(\mathbb{R}, \mathbb{R}^n)$ satisfies $|w(t)| \leq Be^{-bt}$ for some b with $0 < b < a$. We can solve (2.2) with the general solution bounded on $[0, \infty)$ given by

$$x(t) = U(t)P_1 c + U(t) \int_0^t P_1 U(s)^{-1} w(s)\, ds - U(t) \int_t^\infty P_2 U(s)^{-1} w(s)\, ds$$

for $c \in \mathbb{R}^n$. This formula is often referred to as variation of constants. Using property (iii) in Definition 2.1 it is easy to show that the bounded solution so obtained satisfies the same growth behavior at ∞ as w.

As an application of these ideas one can use an exponential dichotomy, variation of constants and the implicit function theorem to prove the stable manifold theorem. See Theorem 4.1 in §13.4 of Coddington and Levinson [5].

In our work with homoclinic solutions we will be concerned with equations as (2.1) where the coefficient matrix approaches a hyperbolic matrix both as $t \to +\infty$ and as $t \to -\infty$. In this case it is not generally true that (2.1) will have an exponential dichotomy on the whole line. For this case we begin with a stronger version of Theorem 2.2.

2.3. LEMMA. *Let* $t \to A(t)$ *be a matrix valued function continuous on* $[0, \infty)$ *and suppose there exists a constant matrix,* A_0*, with Jordan form* J *and a scalar* $b > 0$ *such that* $\sup_{t\geq 0} |A(t) - A_0|e^{2bt} < \infty$. *Then there exists a fundamental solution,* X*, to the differential equation* $\dot{x} = A(t)x$ *such that* $\lim_{t\to\infty} \left[X(t)e^{-tJ} - C\right] e^{at} = 0$ *for some constant matrix* C *and* $a > 0$.

PROOF. Let P be a matrix such that $P^{-1}A_0 P = J$ where J is in Jordan form with the block-diagonal form $J = \operatorname{diag}(J_1, J_2, \cdots, J_r)$. Let k_i denote the order of J_i and λ_i the eigenvalue associated with J_i. We shall assume the Jordan blocks are ordered so that $\Re(\lambda_i) \leq \Re(\lambda_{i+1})$.

Let $y = P^{-1}x$ and $B(t) = P^{-1}A(t)P$. Then the equation $\dot{x} = A(t)x$ becomes

$$\dot{y} = B(t)y = J + (B(t) - J)y. \tag{2.3}$$

We shall construct solutions to this equation relative to each Jordan block. Fix one such block, J_i, and define $p_i = k_1 + k_2 + \cdots + k_{i-1}$ Then J_i occupies columns $p_i + 1$ through $p_i + k_i$. Define q_i so that $\Re(\lambda_{q_i - 1}) < \Re(\lambda_i)$ and $\Re(\lambda_{q_i}) = \Re(\lambda_i)$.

We decompose e^{tJ} by defining $U_1(t)$ and $U_2(t)$ as block diagonal matrices:

$$U_1(t) = \operatorname{diag}(e^{tJ_1}, \cdots, e^{tJ_{q_i - 1}}, 0, \cdots, 0),$$
$$U_2(t) = \operatorname{diag}(0, \cdots, 0, e^{tJ_{q_i}}, \cdots, e^{tJ_r}).$$

We can choose $K > 0$ and a satisfying $0 < 2a \leq b$ such that

$$|U_1(t)| \leq K \exp((\Re(\lambda_i) - 2a)t) \qquad \text{for } t \geq 0,$$
$$|U_2(t)| \leq K \exp((\Re(\lambda_i) - 2a)t) \qquad \text{for } t \leq 0.$$

Now let $\epsilon = b/4K$ and choose $t_0 \geq 0$ such that $|B(t) - J| \leq \epsilon e^{-bt}$ for $t \geq t_0$.

Let e_k denote the kth column of the $n \times n$ identity matrix. For each $j \in \{1, 2, \cdots, k_i\}$, (2.3) can be written

$$y(t) = e^{tJ}\left(e_{p_i+1} + e_{p_i+2} + \cdots + e_{p_i+j}\right) + \int_{t_0}^{t} U_1(t-s)(B(s)-J)y(s)\,ds - \int_{t}^{\infty} U_2(t-s)(B(s)-J)y(s)\,ds.$$

We solve this equation by iteration starting with $y^{(0)} = 0$ and obtaining a sequence, $\{y^{(0)}, y^{(1)}, \cdots\}$, satisfying $|y^{(n)}(t) - y^{(n-1)}(t)| \leq (K_0/2^{n-1}) \exp((\Re(\lambda_i) + 2a)t)$ where K_0 is chosen to satisfy

$$|e^{tJ}(e_{p_i+1} + \cdots + e_{p_i+j})| \leq K_0 \exp((\Re(\lambda_i) + 2a)t).$$

This yields a solution, y_j, to (2.3) satisfying $|y_j(t)| \leq 2K_0 \exp((\Re(\lambda_i) + 2a)t)$ and

$$|y_j(t) - e^{tJ}\left(e_{p_i+1} + e_{p_i+2} + \cdots + e_{p_i+j}\right)| \exp((-\Re(\lambda_i) + 2a)t) \leq K_0.$$

Let $Y_i(t)$ denote the $n \times k_i$ matrix with y_j in column j, let $F_i(t)$ be the $k_i \times k_i$ matrix

$$F_i(t) = \begin{pmatrix} 1 & 1+t & 1+t+\frac{t^2}{2!} & \dots & 1+t+\frac{t^2}{2!}+\cdots+\frac{t^{k_i-1}}{(k_i-1)!} \\ 0 & 1 & 1+t & \dots & 1+t+\frac{t^2}{2!}+\cdots+\frac{t^{k_i-2}}{(k_i-2)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

and let $\hat{F}_i(t)$ be the $n \times k_i$ matrix with $F_i(t)$ in rows $p_i + 1$ through $p_i + k_i$ and all other rows zero. Then we have

$$|Y_i(t) - \hat{F}_i(t)e^{\lambda_i t}|e^{(-\Re(\lambda_i)+2a)t} \leq K_0.$$

Let G_i be the $k_i \times k_i$ matrix with ones on and above the diagonal and zeros below. Then $G_i e^{tJ_i} = F_i(t)e^{\lambda_i t}$. This yields $\lim_{t\to\infty}\left[Y_i(t)e^{-tJ_i} - \hat{G}_i\right]e^{at} = 0$ where $\hat{G}_i$ is the $n \times k_i$ matrix with G_i in rows $p_i + 1$ through $p_i + k_i$ and all other rows zero.

This construction was related to a specific Jordan block J_i. To assemble the results for all blocks we define the $n \times n$ matrix $Y(t)$ by placing $Y_i(t)$ in columns $p_i + 1$ through $p_i + k_i$ for $i = 1, 2, \cdots, r$. In a similar way we combine the matrices $\hat{G}_i$ to get G. Alternatively we have $G = \operatorname{diag}(G_1, \cdots, G_r)$. We now have $\lim_{t\to\infty}\left[Y(t)e^{-tJ} - G\right]e^{at} = 0$.

Finally, we define $X(t) = PY(t)$. Then X satisfies $\dot{X} = A(t)X$ and

$$\lim_{t\to\infty}\left[X(t)e^{-tJ} - PG\right]a^{at} = 0.$$

□

2.4 THEOREM. *Let $t \to A(t)$ be a matrix-valued function continuous for $t \in \mathbb{R}$. Suppose there exists a constant hyperbolic matrix, A_0, and a scalar $b > 0$ such that*

$$\sup_t |A(t) - A_0| e^{b|t|} < \infty.$$

Then there exists a fundamental solution, U, for (2.1) along with positive constants K_0, a and four projections P_{ss}, P_{su}, P_{us}, P_{uu} such that $P_{ss} + P_{su} + P_{us} + P_{uu} = I$ and such that the following hold:

(i) $|U(t)(P_{ss} + P_{us})U(s)^{-1}| \le K_0 e^{2a(s-t)}$ *for* $0 \le s \le t$,

(ii) $|U(t)(P_{uu} + P_{su})U(s)^{-1}| \le K_0 e^{2a(t-s)}$ *for* $0 \le t \le s$.

(iii) $|U(t)(P_{ss} + P_{su})U(s)^{-1}| \le K_0 e^{2a(t-s)}$ *for* $t \le s \le 0$,

(iv) $|U(t)(P_{uu} + P_{us})U(s)^{-1}| \le K_0 e^{2a(s-t)}$ *for* $s \le t \le 0$,

Furthermore, there exists an integer $d \ge 0$ such that $\operatorname{rank}(P_{ss}) = \operatorname{rank}(P_{uu}) = d$.

PROOF. Let J denote the Jordan form for A_0. From Lemma 2.3 there exist two fundamental solutions, U^+ and U^-, for (2.1) a constant matrices C^+, C^- and a constant $a > 0$ such that

$$\sup_{t \ge 0} \left| U^+(t) e^{-tJ} - C^+ \right| e^{2at} < 0,$$
$$\sup_{t \le 0} \left| U^-(t) e^{-tJ} - C^- \right| e^{-2at} < 0.$$

Since these are both fundamental solutions we can write $U^+(t) = U^-(t)R$ for some constant matrix R. We now operate on R by means of elementary column operations. The objective is to obtain $U^+Q = U^-\bar{R}$ with Q upper-triangular and $\bar{R}$ such that the first non-zero entry in each column is one with each column-leading one in a different row.

Suppose we have reached the point where the transformed R has the following property: there exist distinct integers $j_1, j_2, \ldots, j_{s-1}$ such that

$$r_{ij_k} = 0 \quad \text{if } i < k,$$
$$r_{kj_k} = 1,$$
$$r_{ik} = 0 \quad \text{for } 1 \le i < s-1, \quad k \notin \{j_1, j_2, \ldots, j_{s-1}\}.$$

In row s pick the minimum $j_s \notin \{j_1, \ldots, j_{s-1}\}$ such that $p_{sj_s} \neq 0$. Such a j_s must exist as R is non-singular. Now divide column j_s by r_{sj_s} so now $r_{sj_s} = 1$. Next, use column operations to get $r_{sj} = 0$ for $j \notin \{j_1, j_2, \ldots, j_s\}$. Notice we need operate only on columns to the right of column j_s.

Continuing this process through $s = n$ yields a non-singular, upper triangular constant matrix Q such that $U^+(t)Q = U^-(t)\bar{R}$ where $\bar{R}$ has the property that given j, $1 \le j \le n$, there exists $\sigma(j)$ defined by $j_{\sigma(i)} = i$ such that $\sigma(i) \neq \sigma(j)$ for $i \neq j$, $\bar{r}_{ij} = 0$ for $i < \sigma(j)$, and $\bar{r}_{\sigma(j),j} = 1$.

Define $U(t) = U^+(t)Q = U^-(t)\bar{R}$ and define four projection matrices with all zero entries except as follows:

$$(P_{ss})_{ii} = 1 \text{ if } i \le d_s \text{ and } \sigma(i) > d_s,$$
$$(P_{us})_{ii} = 1 \text{ if } i \le d_s \text{ and } \sigma(i) \le d_s,$$
$$(P_{su})_{ii} = 1 \text{ if } i > d_s \text{ and } \sigma(i) > d_s,$$
$$(P_{uu})_{ii} = 1 \text{ if } i > d_s \text{ and } \sigma(i) \le d_s.$$

Since Q is upper triangular we can write

$$Q = \begin{pmatrix} Q_{11} & Q_{12} \\ 0 & Q_{22} \end{pmatrix} \qquad \text{and} \qquad Q^{-1} = \begin{pmatrix} Q_{11}^{-1} & -Q_{11}^{-1}Q_{12}Q_{22}^{-1} \\ 0 & Q_{22}^{-1} \end{pmatrix}$$

where Q_{11} is a $d_s \times d_s$ submatrix. We also have

$$P_{ss} + P_{us} = \begin{pmatrix} I_s & 0 \\ 0 & 0 \end{pmatrix}.$$

These results yield

$$\begin{aligned} U(t)(P_{ss} + P_{us})U(s)^{-1} &= U^{+}(t)Q(P_{ss} + P_{us})Q^{-1}U^{-}(s)^{-1} \\ &= U^{+}(t)\mathrm{e}^{-tJ} \begin{pmatrix} \mathrm{e}^{tJ_1} & 0 \\ 0 & \mathrm{e}^{tJ_2} \end{pmatrix} \begin{pmatrix} I_s & -Q_{12}Q_{22}^{-1} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mathrm{e}^{-sJ_1} & 0 \\ 0 & \mathrm{e}^{-sJ_2} \end{pmatrix} \mathrm{e}^{sJ}U^{+}(s)^{-1} \\ &= \left[U^{+}(t)\mathrm{e}^{-tJ}\right] \begin{pmatrix} \mathrm{e}^{(t-s)J_1} & -\mathrm{e}^{tJ_1}Q_{12}Q_{22}^{-1}\mathrm{e}^{-sJ_2} \\ 0 & 0 \end{pmatrix} \left[\mathrm{e}^{sJ}U^{+}(s)^{-1}\right]. \end{aligned}$$

The expressions in square brackets are bounded for $t \geq 0$, $s \geq 0$. We can choose K_1 such that $|\mathrm{e}^{(t-s)J_1}| \leq K_1\mathrm{e}^{-2M(t-s)} = K_1\mathrm{e}^{2M(s-t)}$ when $t - s \geq 0$. In addition, for $t \geq 0$ we can find K_2 such that $|\mathrm{e}^{tJ_1}| \leq K_2\mathrm{e}^{-2Mt}$ and $|\mathrm{e}^{-tJ_2}| \leq K_2\mathrm{e}^{-2Mt} \leq K_2\mathrm{e}^{2Mt}$. This proves (i). Part (ii) follows in a similar manner using

$$P_{su} + P_{uu} = \begin{pmatrix} 0 & 0 \\ 0 & I_u \end{pmatrix}.$$

We now turn to (iii). If we interchange columns of $\bar{R}$ so that column j moves to column $\sigma(j)$ the result is a matrix with zeros above the diagonal. In terms of matrices there exists W such that $\hat{R} = \bar{R}W$ is lower-triangular. The matrix $P_{ss} + P_{su}$ consists of ones on the diagonal precisely when $\sigma(j) > d_s$. This means that $W^{-1}(P_{ss} + P_{su})W = \begin{pmatrix} 0 & 0 \\ 0 & I_u \end{pmatrix}$.

Combining these results yields

$$U(t)(P_{ss} + P_{su})U(s)^{-1} = \left[U^{-}(t)\mathrm{e}^{-tJ}\right]\mathrm{e}^{tJ}\hat{R}\begin{pmatrix} 0 & 0 \\ 0 & I_u \end{pmatrix}\hat{R}^{-1}\mathrm{e}^{-sJ}\left[\mathrm{e}^{sJ}U^{-}(s)^{-1}\right].$$

Part (iii) follows from this; a similar argument yields (iv). □

Let $A(t)$ be as in Theorem 2.4 and consider the equation

$$\dot{x} = A(t)x + w. \tag{2.4}$$

We shall be interested in locating homoclinic solutions and so we use a from Theorem 2.4 and work in the Banach space

$$\mathbb{Z} = \left\{ \mathcal{C}^0(\mathbb{R}, \mathbb{R}^n) \,\middle|\, \sup_t |z(t)|e^{a|t|} < \infty \right\}.$$

However, (2.4) with $w \in \mathbb{Z}$ cannot be solved for $x \in \mathbb{Z}$. Rather, to solve (2.4) for $x \in \mathbb{Z}$ we must take $w \in \bar{\mathbb{Z}}$ where

$$\bar{\mathbb{Z}} = \left\{ z \in \mathbb{Z} \;\middle|\; \int_{-\infty}^{\infty} P_{uu} U(t)^{-1} z(t)\, dt = 0 \right\}.$$

The space $\bar{\mathbb{Z}}$ can be characterized in another way. The expression $U^{-1}z$ is the inner product of z with a rows of U^{-1} or, equivalently, with the columns of $U^{\perp}$, the adjoint. Furthermore, $P_{uu}Uz$ uses only the columns of $U^{\perp}$ which are bounded functions for all t. These can be obtained directly from the adjoint equation $\dot{v} = -A(t)^t v$.

Now let $w \in \bar{\mathbb{Z}}$. Then the general solution in $\mathbb{Z}$ to (2.4) is $z = UP_{ss}\beta + Kw$ where β is an arbitrary constant vector and K is a variation of constants map defined by

$$\begin{aligned}(Kw)(t) &= U(t)\left[\left(\int_0^t P_{ss} + \int_{-\infty}^t P_{us} - \int_t^{\infty} P_{su} + \int_{-\infty}^t P_{uu}\right) U(s)^{-1} w(s)\, ds\right] \\ &= U(t)\left[\left(\int_0^t P_{ss} + \int_{-\infty}^t P_{us} - \int_t^{\infty} P_{su} - \int_t^{\infty} P_{uu}\right) U(s)^{-1} w(s)\, ds\right].\end{aligned}$$

We can give an interesting geometric interpretation of these ideas which will be useful later. Consider a nonlinear differential equation

$$\dot{x} = f(x,t) \tag{2.5}$$

with $x \in \mathbb{R}^n$ and suppose the following hold:

(i) f is periodic in t with period T.
(ii) $x = 0$ is a hyperbolic equilibrium.
(iii) There exists a homoclinic solution γ. That is, $\dot{\gamma}(t) = f(\gamma(t))$ and

$$\lim_{t \to -\infty} \gamma(t) = \lim_{t \to +\infty} \gamma(t) = 0.$$

We can view this equation as autonomous in $\mathbb{R}^{n+1}$ and in this space consider the stable and unstable manifolds of the origin. These manifolds intersect along the homoclinic orbit and one often needs to show that this intersection is transverse. Equivalently, since the vector field is periodic one can look at the period map of the flow and the corresponding invariant manifolds in $\mathbb{R}^n$. Questions of transversality are concerned with the tangent spaces to the manifolds and the analytic expression of this uses solutions to the variational equation.

By the variational equation along γ we mean the linear differential equation

$$\dot{u} = D_1 f(\gamma(t), t) u. \tag{2.6}$$

The limiting form of (2.6) as $t \to +\infty$ is $\dot{u} = D_1(0,t)u$ and this limit is approached at an exponential rate. The question to be answered is whether (2.6) has an exponential dichotomy for the whole line.

2.5 THEOREM. *The following are equivalent:*

(i) *The stable and unstable manifolds for the flow of (2.5) intersect transversely in* $\mathbb{R}^{n+1}$.
(ii) *The stable and unstable manifolds for the period map of (2.5) intersect transversely in* $\mathbb{R}^n$.
(iii) *The variational equation (2.6) has no nontrivial solution bounded for all* t.
(iv) *The variational equation (2.6) has an exponential dichotomy on the whole line.*

If these conditions hold we say that (2.5) has a transverse homoclinic orbit.

Symbolic Dynamics and Differential Equations

In his 1967 paper, Smale [27] developed the concept of the Bernoulli shift and then used a construction now referred to as the Smale horseshoe to embed the shift map into the discrete flow of a differentiable map. This idea has been extended in a large body of work to the flow of a differential equation by using the Poincare map or the period map for a periodic vector field. A couple of references are [25] and [29].

An alternate approach, using exponential dichotomies, was introduced by K. Palmer in [26] where he proved the following result.

2.6 THEOREM. *Suppose that (2.5) has a transverse homoclinic orbit. Then given N there exists a topological conjugacy between Σ_N and an iterate of the period map of (2.5). In other words, there exists a homeomorphism, ϕ, of Σ_N onto a compact subset of $\mathbb{R}^n$ on which the $2m$th iterate F^{2m} of the period map is invariant and satisfies $F^{2m} \circ \phi = \phi \circ \varphi$ where φ is the Bernoulli shift on Σ_N.*

OUTLINE OF PROOF. Recall that the vector field is periodic with period T. We choose a small neighborhood, U, of the origin in $\mathbb{R}^n$ and then choose an integer, m, sufficiently large so that the points $\{\gamma(\pm mT), \gamma(\pm(m-1)T), \dots, \gamma(\pm(m+N-1)T)\}$ lie in U. We now consider N arcs which follow the homoclinic orbit for $-(m-k)T \le t \le (m+k)T$ for $k = 0, 1, \dots, N-1$. These arcs all have their endpoints in U.

Fix $\sigma \in \Sigma_N$. We define a piecewise continuous path, v_σ, in $\mathbb{R}^n$ by

$$v_\sigma(t) = \gamma(t - (2k-1)mT + \sigma_k T), \quad 2kmT \le t \le 2(k+1)mT.$$

This path travels around the homoclinic orbit, jumping from one arc to another with the sequence of arcs determined by σ and all the jumps occurring within U. Since v_σ follows a solution to (2.5) while on an arc and since the size of the jumps is determined by U, we can choose U small enough to make v_σ an arbitrarily accurate pseudorbit.

The main lemma in Palmer's work can now be paraphrased as follows: By hypothesis the variational equation along γ has an exponential dichotomy on the whole line. The path v_σ can be made arbitrarily close to the arcs of γ. From this it follows that there exists an exponential dichotomy, (V, P_1, P_2), along v_σ.

In (2.5) we now make the change of variable $x = v_\sigma + z$. The equation for z is

$$\dot{z}(t) = D_1 f(v_\sigma(t), t) z(t) + g(z(t), t)$$

where

$$g(z,t) = f(v_\sigma(t), t) - f(v_\sigma(t), t) - D_1 f(v_\sigma(t), t) + f(v_\sigma(t), t) - \dot{v}_\sigma(t).$$

Let $\mathbb{Z}$ denote the Banach space of continuous, bounded function from $\mathbb{R}$ to $\mathbb{R}^n$. Using the fact that the variational equation along v_σ has an exponential dichotomy we can use variation of parameter to define the differentiable function $F : \mathbb{Z} \to \mathbb{Z}$ by

$$F(z)(t) = V(t) \int_{-\infty}^{t} P_1 V(s)^{-1} g(z(s), s)\, ds - V(t) \int_{t}^{\infty} P_2 V(s)^{-1} g(z(s), s)\, ds.$$

This function has the following properties:

- $F(z) = 0$ implies $z + v_\sigma$ is a solution to (2.5) which is close to v_σ.
- $|F(0)|$ is small due to the fact that v_σ is a pseudorbit.
- $\mathrm{DF}(0) = 0$.

These properties allow the use of the contraction mapping theorem to find z_σ with $F(z_\sigma) = 0$ and then $x_\sigma = v_\sigma + z_\sigma$ is a solution to (2.5) which shadows v_σ. The homeomorphism is now given by $\phi(\sigma) = x_\sigma(0)$. Some details of this last step are illustrated in the proof of Theorem 4.3 below. □

CHAPTER III

HOMOCLINIC BIFURCATIONS AND MELNIKOV THEORY FOR ORDINARY DIFFERENTIAL EQUATIONS

In the preceding chapter we reduced the problem of showing the existence of chaos for an ordinary differential equation to that of showing the existence of a transverse homoclinic orbit. The latter is the subject of what is often called Melnikov theory. A seminal paper in this field was that of Holmes [17]. Although his problem is ostensibly a partial differential equation, Holmes uses a formal Bubnov-Galerkin-Kantorovich reduction to an ordinary differential equation in $\mathbb{R}^2$. For this problem he then uses the original results of Melnikov [23]. Subsequently, Holmes, with J. Marsden, provided a rigorous treatment of the partial differential equation[18], replacing the Bubnov-Galerkin-Kantorovich projection with semigroup theory.

The approach of Melnikov is geometric and this flavor is continued by Holmes and subsequently by Wiggins [29] and others. Immediately after Holmes' 1979 paper an analytical approach to the subject, using exponential dichotomies, was initiated by Chow, Hale and Mallet-Paret [4] and this flavor has been continued by Palmer in [26] as well as other work. The first extension of Melnikov theory to higher dimension was my thesis [10] which followed the geometric approach. My subsequent work has gravitated to the analytical school [11], [12], [13]. What follows here will continue in this style.

As a model problem to fix the ideas consider the forced, damped Duffing equation with negative stiffness

$$\ddot{x} = x - 2x^3 + \mu_2 \dot{x} + \mu_1 \cos \omega t. \tag{3.1}$$

We consider (3.1) as a first order system in $\mathbb{R}^2$ and view the μ_i terms as perturbations. The unperturbed equation, $\ddot{x} = x - 2x^3$, is autonomous, has a hyperbolic equilibrium at $x = 0$, and has a homoclinic solution given by $\gamma(t) = \operatorname{sech} t$. The idea is to find a region in parameter space, i.e. the μ_1-μ_2 plane, where the perturbed equation has a transverse homoclinic solution. One can then apply Theorem 2.6. We return to this in Example 3.1 below.

The Existence of Homoclinic Solutions

We consider equations of the form

$$\dot{x} = f(x, \mu, t) = f_0(x) + \mu_1 f_1(x, \mu, t) + \mu_2 f_2(x, \mu, t) \tag{3.2}$$

with $x \in \mathbb{R}^n$. We make the following the assumptions for (3.2):

(i) Each f_i is $\mathcal{C}^2$ in all arguments.

(ii) $f_0(0) = 0$ and the eigenvalues of $Df_0(0)$ lie off the imaginary axis.

(iii) There exists a nontrivial homoclinic solution γ for the unperturbed equation. That is, $\gamma \neq 0$, $\dot{\gamma}(t) = f_0(\gamma(t))$ and $\lim_{t\to-\infty} \gamma(t) = \lim_{t\to+\infty} \gamma(t) = 0$.

(iv) f is periodic in t.

Application of Theorem 2.4 to the variational equation $\dot{u} = Df_0(\gamma)u$ yields a fundamental solution U, an integer d, constants K_0, a and four projections P_{ss}, P_{su}, P_{us}, P_{uu}. By renumbering we can assume

$$P_{ss} = \begin{bmatrix} I_d & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad P_{uu} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I_d & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where I_d is the $d \times d$ identity matrix and also, in column d of U, $u_d = \dot{\gamma}$.

The integer d has a geometric interpretation. Let W^s, W^u denote the stable, unstable manifolds respectively of the origin for the unperturbed equation $\dot{x} = f_0(x)$ and let $P = \gamma(0)$. Then $P \in W^s \cup W^u$ and $d = T_P W^s \cap T_P W^u$, the dimension of intersection of the invariant manifolds.

Let u_j denote column j of U. Note that $\{u_1, \dots, u_d\}$ form a basis for the vector space of bounded solutions to the variational equation and that any such basis can be extended to a fundamental solution U as in Theorem 2.4.

Now consider the adjoint matrix $U^\perp = (U^{-1})^t$. This matrix is a fundamental solution to the adjoint equation $\dot{v} = -Df_0(\gamma)^t v$. If we let $u_j^\perp$ denote column j of $U^\perp$ or, equivalently, row j of U^{-1} then $\{u_{d+1}^\perp, \dots, u_{2d}^\perp\}$ forms a basis for the vector space of bounded solutions to the adjoint equation.

We now have the following result from [12].

3.1 THEOREM. *Let $\{u_1, \dots, u_{d-1}, \dot{\gamma}\}$ be a basis for the vector space of bounded solutions to the variational equation $\dot{u} = Df_0(\gamma)u$ and let $\{v_1, \dots, v_d\}$ be a basis for the vector space of bounded solutions to the adjoint equation $\dot{v} = -Df_0(\gamma)^t v$. There exist a connected open set $V \subset \mathbb{R}^2 \times \mathbb{R}^{d-1}$ with $(0,0) \in V$ and C^2 functions $H : V \times \mathbb{R} \to \mathbb{R}^d$ and $\Gamma : V \times \mathbb{R} \to \mathbb{Z}$ denoted $H(\mu, \beta, \alpha)$, $\Gamma(\mu, \beta, \alpha)$ with the following properties:*

(i) *If $H(\mu, \beta, \alpha) = 0$ then $\Gamma(\mu, \beta, \alpha)$ is a homoclinic solution to (3.2),*

(ii) $\Gamma(0, 0, \alpha) = \gamma$,

(iii) $\dfrac{\partial \Gamma}{\partial \beta_k}(0, 0, \alpha) = u_k$,

(iv) $H(0, 0, \alpha) = 0$,

(v) $\dfrac{\partial H_i}{\partial \mu_j}(0, 0, \alpha) = \displaystyle\int_{-\infty}^{\infty} \langle v_i(t), f_j(\gamma(t), 0, t + \alpha) \rangle \, dt$,

(vi) $\dfrac{\partial H_i}{\partial \beta_j}(0, 0, \alpha) = 0$,

(vii) $\dfrac{\partial^2 H_i}{\partial \beta_j \partial \beta_k}(0, 0, \alpha) = \displaystyle\int_{-\infty}^{\infty} \left\langle v_i, D^2 f_0(\gamma) u_j u_k \right\rangle \, dt$.

OUTLINE OF PROOF. To find a homoclinic solution for (3.2) which is close to γ, the homoclinic solution for the unperturbed equation, we make the change of variable $x(t) = \gamma(t - \alpha) + z(t - \alpha)$. The equation for z is

$$\dot{z} = Df_0(\gamma)z + G(z, \mu, \alpha)(t) \tag{3.3}$$

where G is given by

$$G(z,\mu,\alpha)(t) = f_0(\gamma(t)+z(t))+\mu_1 f_1(\gamma(t)+z(t),\mu,t+\alpha)+\mu_2 f_2(\gamma(t)+z(t),\mu,t+\alpha).$$

To begin, let us assume $f_i(0,\mu,t)=0$ in (3.2). Then we have $G:\mathbb{Z}\times\mathbb{R}^2\times\mathbb{R}\to\mathbb{Z}$. The important point is that G is quadratic in z.

Apply Theorem 2.4 to the variational equation $\dot{u} = Df_0(\gamma)u$ to obtain U, constants K_0, a, d and four projections. Then using a define the Banach spaces $\mathbb{Z}$, $\bar{\mathbb{Z}}$ as on pp. 12-13. We seek a solution to (3.3) in $\mathbb{Z}$ which we obtain by the method of Lyapunov-Schmidt. We first let $\Phi:\mathbb{R}\to\mathbb{R}$ be a smooth function satisfying

$$\sup_t \left|\Phi(t)U(t)e^{a|t|}\right| < \infty, \qquad \int_{-\infty}^{\infty}\Phi(t)\,dt = 1.$$

Then use Φ to define the projection $\Pi:\mathbb{Z}\to\mathbb{Z}$ as

$$(\Pi z)(t) = \Phi(t)U(t)\int_{-\infty}^{\infty} P_{uu}U(s)^{-1}z(s)\,ds.$$

It is easy to check that $\mathrm{Im}(I-\Pi)=\bar{\mathbb{Z}}$.

Equation (3.3) now splits into two parts

$$\dot{z} = Df_0(\gamma)z + (I-\Pi)G(z,\mu,\alpha), \tag{3.4a}$$

$$\Pi G(z,\mu,\alpha) = 0. \tag{3.4b}$$

We convert (3.4a) to an integral equation by defining $F:\mathbb{Z}\times\mathbb{R}^2\times\mathbb{R}^{d-1}\times\mathbb{R}\to\mathbb{Z}$ by

$$F(z,\mu,\beta,\alpha) = \sum_{i=1}^{d-1}\beta_i\mu_i + K(I-\Pi)G(z,\mu,\alpha)$$

where K is the variation of constants map defined on p. 13. Note the range of the sum here. By omitting $i=d$ we exclude solutions which vary from each other by only a phase shift.

To obtain a solution to (3.4a) in $\mathbb{Z}$ we use the contraction mapping theorem to find a fixed point for F. This yields an open set $V\subset\mathbb{R}^2\times\mathbb{R}^{d-1}$ containing the origin and a function $\psi:V\times\mathbb{R}\to\mathbb{Z}$ with the following properties:

$$\begin{aligned}
\psi(\mu,\beta,\alpha) &= \sum_{i=1}^{d-1}\beta_i\mu_i + K(I-\Pi)G(\psi(\mu,\alpha,\beta),\mu,\alpha),\\
\psi(0,o,\alpha) &= 0,\\
\frac{\partial\psi}{\partial\beta_i}(0,0,\alpha) &= u_j.
\end{aligned}$$

Our bifurcation equation is $\Pi G(\psi(\mu,\beta,\alpha),\mu,\alpha)=0$ or, in an equivalent form, $H(\mu,\beta,\alpha)=0$ where

$$H_i(\mu,\beta,\alpha) = \int_{-\infty}^{\infty}\left\langle u_{i+d}^{\perp}, G(\psi(\mu,\beta,\alpha),\mu,\alpha)\right\rangle\,dt.$$

By a linear change of variables we can replace $\{u_{d+1}^{\perp}, \dots, u_{2d}^{\perp}\}$ with $\{v_1, \dots, v_d\}$.

In the case where $f_i(0, \mu, t) \neq 0$ in (3.2) we can first obtain a small bounded solution, x_0, which is $O(|\mu|)$ and then find a solution which is homoclinic to x_0. For details see Theorem 14 of [12]. □

The preceding results lead us to define, for $\mu \in \mathbb{R}^2$, $\beta \in \mathbb{R}^{d-1}$, $\alpha \in \mathbb{R}$,

$$a_{ij}(\alpha) = \int_{-\infty}^{\infty} \langle v_i, f_j(\gamma(t), 0, t + \alpha) \rangle \, dt \quad \begin{cases} 1 \leq i \leq d \\ 1 \leq j \leq 2, \end{cases}$$

$$b_{ijk} = \int_{-\infty}^{\infty} \langle v_i, D^2 f_0(\gamma) u_j u_k \rangle \, dt \quad \begin{cases} 1 \leq i \leq d \\ 1 \leq j, k \leq d - 1, \end{cases}$$

$$M_i(\mu, \beta, \alpha) = \sum_{j=1}^{2} a_{ij}(\alpha) \mu_j + \frac{1}{2} \sum_{j,k=1}^{d-1} b_{ijk} \beta_j \beta_k, \quad 1 \leq i \leq d. \tag{3.5}$$

The conditions for the existence of a homoclinic solution to (3.2) now become

$$H(\mu, \beta, \alpha) = M(\mu, \beta, \alpha) + h.o.t. = 0.$$

If $M(\mu_0, \beta_0, \alpha_0) = 0$ and $D_{(\beta,\alpha)} M(\mu_0, \beta_0, \alpha_0)$ is nonsingular then we can use the implicit function theorem to get $H = 0$ for (μ, β, α) near $(\mu_0, \beta_0, \alpha_0)$. We return to this below.

To apply Theorem 2.6 it is necessary to show, in addition, that the homoclinic orbit obtained in Theorem 3.1 is transverse. This is accomplished by showing the variational equation along the homoclinic orbit when $\mu \neq 0$ has no bounded solution and using Theorem 2.5. It turns out that the condition $D_{(\beta,\alpha)} M(\mu_0, \beta_0, \alpha_0)$ nonsingular mentioned in the preceding paragraph also yields transversality of the orbit. For details see [13]. We now have the following result.

3.2 THEOREM. *Let M be as in (3.5). If $M(\mu_0, \beta_0, \alpha_0) = 0$ and if in addition $D_{(\beta,\alpha)} M(\mu_0, \beta_0, \alpha_0)$ is nonsingular then there exists $\xi_0 > 0$ such that when $0 < \xi < \xi_0$ the differential equation $\dot{x} = f(x, \xi\mu_0, t)$ has a transverse homoclinic orbit γ_ξ.*

Let us look at some special cases of the preceding theorem. If $x \in \mathbb{R}^2$ we must have $d = 1$ and $u_1 = \dot{\gamma}$. Denoting $\gamma = (\gamma_1, \gamma_2)$ we can take

$$v_1 = (-\dot{\gamma}_2(t), \dot{\gamma}_1(t)) \exp\left(- \int_0^t (\nabla \cdot f_0)(\gamma(s)) \, ds \right).$$

There is no β and the condition for a homoclinic solution is the scalar equation $M(\mu, \alpha) = a_1(\alpha)\mu_1 + a_2(\alpha)\mu_2 = 0$. This yields the following well known result. For comparison see Theorem 3.3, p.382 of [3] and Theorem 4.5.3 of [15].

3.3. COROLLARY. *Let $\dot{x} = f(x, \mu, t)$ be as in (3.2) with $x \in \mathbb{R}^2$ and let*

$$a_j(\alpha) = \int_{-\infty}^{\infty} \det(\dot{\gamma}(t), f_j(\gamma(t), 0, t + \alpha)) \exp\left(- \int_0^t (\nabla \cdot f_0)(\gamma(s)) \, ds \right) dt$$

for $j = 1, 2$. If $a_1(\alpha_0)\mu_{0,1} + a_2(\alpha_0)\mu_{0,2} = 0$ and $a_1'(\alpha_0)\mu_{0,1} + a_2'(\alpha_0)\mu_{0,2} \neq 0$ then there exists $\xi_0 > 0$ such that the differential equation $\dot{x} = f(x, \xi\mu_0, t)$ has a transverse homoclinic solution for $0 < \xi \leq \xi_0$.

The preceding result generalizes with little difficulty to higher n as long as we have $d = 1$, the case studied by Palmer [26]. The difference between $n = 2$ and $n > 2$ is that one must compute v_1 directly.

3.4. COROLLARY. *Let* $\dot{x} = f(x, \mu, t)$ *be as in (3.2) with* $d = 1$. *Let* v *be a nontrivial bounded solution to the variational equation and for* $j = 1, 2$ *define* $a_j(\alpha) = \int_{-\infty}^{\infty} \langle v(t), f_j(\gamma(t), 0, t + \alpha)\rangle \, dt$. *If* $a_1(\alpha_0)\mu_{0,1} + a_2(\alpha_0)\mu_{0,2} = 0$ *and* $a_1'(\alpha_0)\mu_{0,1} + a_2'(\alpha_0)\mu_{0,2} \neq 0$ *then there exists* $\xi_0 > 0$ *such that the differential equation* $\dot{x} = f(x, \xi\mu_0, t)$ *has a transverse homoclinic solution for* $0 < \xi \leq \xi_0$.

Case of a manifold of homoclinic orbits. Suppose that $W^s \cap W^u$ has a connected component which is a manifold of dimension d and which contains the orbit of γ. Then in (3.5) all $b_{ijk} = 0$, the hypotheses of Theorem 3.2 cannot be satisfied and an alternate bifurcation function is required. This case arises in certain integrable Hamiltonian systems [21].

Let W^h denote a homoclinic d-manifold containing γ, let U_0 be an open neighborhood of the origin in $\mathbb{R}^{d-1}$, let $\eta : U_0 \to W^h$ be a differentiable function denoted $\beta \to \eta(\beta)$ with $\eta(0) = P$, let $t \to \gamma_\beta(t)$ be the solution to the unperturbed equation (3.2) satisfying $\gamma_\beta(0) = \eta(\beta)$, and assume η is constructed so that $(\beta, t) \to \gamma_\beta(t)$ establishes local coordinates on W^h. In other words, the original orbit γ is embedded in a $(d-1)$-parameter family of homoclinic orbits.

For each fixed β we let $\{v_{\beta 1}, \ldots, v_{\beta d}\}$ denote a basis for the vector space of bounded solutions to the adjoint equation $\dot{v} = -D_1 f_0(\gamma_\beta, 0)^t v$. Without loss of generality we can assume that each $v_{\beta i}$ depends differentially on β. Now define functions $a_{ij} : U_0 \times \mathbb{R} \to \mathbb{R}$ and $M : \mathbb{R}^2 \times U_0 \times \mathbb{R} \to \mathbb{R}^d$ by

$$a_{ij}(\beta, \alpha) = \int_{-\infty}^{\infty} \langle v_{\beta i}(t), f_j(\gamma_\beta(t), 0, 0, t + \alpha)\rangle \, dt, \qquad \begin{cases} 1 \leq i \leq d \\ 1 \leq j \leq 2, \end{cases}$$

$$M_i(\mu, \beta, \alpha) = \sum_{j=1}^{2} a_{ij}(\beta, \alpha)\mu_j, \quad 1 \leq i \leq d. \tag{3.6}$$

We can now replace (3.5) with (3.6) in Theorem 3.2

3.5 THEOREM. *Let* M *be as in (3.6). If* $M(\mu_0, \beta_0, \alpha_0) = 0$ *and if in addition* $D_{(\beta,\alpha)}M(\mu_0, \beta_0, \alpha_0)$ *is nonsingular then there exists* $\xi_0 > 0$ *such that when* $0 < \xi < \xi_0$ *the differential equation* $\dot{x} = f(x, \xi\mu_0, t)$ *has a transverse homoclinic orbit* γ_ξ.

Examples

We now illustrate the theory with examples. To simplify some of the formulas we denote $r(t) = \operatorname{sech} t$. Note that $\ddot{r} = r - 2r^3$ and $\dddot{r} = (1 - 6r^2)\dot{r}$.

Example 3.1. Consider the equation

$$\ddot{x} = x - 2x^3 - \mu_2\dot{x} + \mu_1 \cos \omega t. \tag{3.7}$$

This equation, referred to as the forced, damped Duffing equation with negative stiffness, is an extremely prevalent example of chaos in the ordinary differential equation literature. As mentioned previously this equation is the principle object of study in the pioneering work of Holmes [17] and it appears also in his textbook [15].

The unperturbed equation is $\ddot{x} = x - 2x^3$ for which we must find a solution homoclinic to the origin. Multiplying by $\dot{x}$, integrating with respect to t and multiplying by 2 yields $\dot{x}^2 = x^2 - x^4$ and then $\dot{x} = \pm x\sqrt{1 - x^2}$. We have taken the constant of integration zero to get a solution through the origin.

By separating variables and integrating the preceding equation we can obtain two homoclinic solution. We have

$$t = \int \frac{\pm 1}{x\sqrt{1-x^2}}\, dx = \mp \operatorname{sech}^{-1} x$$

so that we can take $x = \pm r(t)$. Using a constant of integration here will yield time translates of these results.

We use $x_1 = x$, $x_2 = \dot{x}$ to put (3.7) in first order form and then apply Corollary 3.3. Using the homoclinic solution $\gamma = (r, \dot{r})$ we compute

$$a_1(\alpha) = \int_{-\infty}^{\infty} \dot{r} \cos \omega(t+\alpha)\, dt = \pi\omega \operatorname{sech} \tfrac{\pi\omega}{2} \sin \omega\alpha,$$
$$a_2(\alpha) = -\int_{-\infty}^{\infty} \dot{r}^2\, dt = -\tfrac{2}{3}$$

and then we must solve

$$M(\mu, \alpha) = \left(\pi\omega \operatorname{sech} \tfrac{\pi\omega}{2} \sin \omega\alpha\right) \mu_1 - \tfrac{2}{3}\mu_2 = 0.$$

By varying α we get a solution to this equation when $|\mu_2/\mu_1| \leq m_0$ where $m_0 = \frac{3}{2}\pi\omega \operatorname{sech} \frac{\pi\omega}{2}$.

Thus for fixed, sufficiently small forcing (μ_1) there exists a range of damping values (μ_2) for which (3.7) has a transverse homoclinic solution (Corollary 3.3) and hence exhibits chaos (Theorem 2.6). Note that for given forcing, if the damping is large enough the chaos will not appear. This is summarized in Fig. 3.1

FIGURE 3.1. Chaotic Region for (3.7)

Exercise. Chow, Hale and Mallet-Paret [4] use the example

$$\ddot{x} = 4x - 6x^2 - \mu_2\dot{x} + \mu_1 \cos \omega t.$$

Find the conditions for chaos. The coefficients have been scaled for convenience.

Example 3.2. As a generalization of the preceding example consider

$$\begin{aligned}\ddot{x} &= x - 2x(x^2+y^2+z^2) - \mu_2\dot{x} + \mu_1\cos\omega t,\\ \dot{y} &= y - 2y(x^2+y^2+z^2) + \mu_1\cos\omega t,\\ \dot{z} &= z - 2z(x^2+y^2+z^2) + \mu_1\cos\omega t.\end{aligned}$$

Let $x_1 = x$, $x_2 = \dot{x}$, $x_3 = y$, $x_4 = z$ to get

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= x_1 - 2x_1(x_1^2+x_3^2+x_4^2) - \mu_2 x_2 + \mu_1\cos\omega t,\\ \dot{x}_3 &= -x_3 - 2x_3(x_1^2+x_3^2+x_4^2) + \mu_1\cos\omega t,\\ \dot{x}_4 &= -x_4 - 2x_4(x_1^2+x_3^2+x_4^2) + \mu_1\cos\omega t.\end{aligned}$$

When $\mu_1 = \mu_2 = 0$ we have a homoclinic solution given by $x = r(t)$, $y = 0$, $z = 0$ or, in first order form, $\gamma = (r, \dot{r}, 0, 0)$. The eigenvalues at the origin for the unperturbed equation are $\{-1, +1, +1, +1\}$ from which we know that $\dim(W^s) = 1$, $d = 1$, the only nontrivial bounded solution to the variational equation is $u_1 = \dot{\gamma}$ (up to translates)and we need only one nontrivial bounded solution to the adjoint equation.

If we let $v = (-\dot{\eta}, \eta, \zeta, \varphi)$ the adjoint equation becomes

$$\ddot{\eta} = (1-6r^2)\eta, \qquad \dot{\zeta} = (1+2r^2)\zeta, \qquad \dot{\varphi} = (1+2r^2)\varphi$$

and a bounded solution is given by $v_1 = (-\ddot{r}, \dot{r}, 0, 0)$. We now apply Corollary 3.4 and Theorem 2.6 obtaining the same answer as in Example 1.

Example 3.3. Consider the equations

$$\begin{aligned}\ddot{x} &= x - x^3 - xy^2 - \mu_2\dot{x} + \mu_1 A\cos\omega t,\\ \ddot{y} &= y - \tfrac{4}{3}y^3 - \tfrac{2}{3}x^3 - \mu_2\dot{y} + \mu_1 B\cos\omega t.\end{aligned}$$

We work in the phase space $(x, \dot{x}, y, \dot{y})$ and find the eigenvalues at the origin for the unperturbed equations to be $\{-1, -1, +1, +1\}$ so that $\dim(W^s) = \dim(W^u) = 2$.

The unperturbed equation has a homoclinic solution $x = y = r$ or in phase space $\gamma = (r, \dot{r}, r, \dot{r})$. If we denote a solution to the variational equation by $u = (\eta, \dot{\eta}, \zeta, \dot{\zeta})$ then we must solve

$$\ddot{\eta} = (1-4r^2)\eta - 2r^2\zeta, \qquad \ddot{\zeta} = -2r^2\eta + (1-4r^2)\zeta.$$

We can take $\eta = -\zeta = r$ or $\eta = \zeta = \dot{r}$. Thus we get two bounded solutions

$$\begin{aligned}u_1 &= (r, \dot{r}, -r, -\dot{r}),\\ u_2 &= \dot{\gamma} = (\dot{r}, \ddot{r}, \dot{r}, \ddot{r}).\end{aligned}$$

This shows that $d = 2$.

To solve the adjoint equation we denote $v = (-\dot{\phi}, \phi, -\dot{\psi}, \psi)$ and solve

$$\ddot{\phi} = (1-4r^2)\phi - 2r^2\psi, \qquad \ddot{\psi} = -2r^2\phi + (1-4r^2)\psi.$$

For bounded solutions we can take $\phi = -\psi = r$ and $\phi = \psi = \dot{r}$ so we get

$$v_1 = (-\dot{r}, r, \dot{r}, -r),$$
$$v_2 = (-\ddot{r}, \dot{r}, -\ddot{r}, \dot{r}).$$

In the notation of Theorem 3.2 we have

$$a_{11}(\alpha) = \int_{-\infty}^{\infty} r[A\cos\omega(t+\alpha)] - r[B\cos\omega(t+\alpha)]\,dt = (A-B)\pi \operatorname{sech}\tfrac{\pi\omega}{2}\cos\omega\alpha,$$
$$a_{12}(\alpha) = \int_{-\infty}^{\infty} r(-\dot{r}) - r(-\dot{r})\,dt = 0,$$
$$a_{21}(\alpha) = \int_{-\infty}^{\infty} \dot{r}[A\cos\omega(t+\alpha)] + \dot{r}[B\cos\omega(t+\alpha)]\,dt = (A+B)\pi\omega \operatorname{sech}\tfrac{\pi\omega}{2}\sin\omega\alpha,$$
$$a_{22}(\alpha) = \int_{-\infty}^{\infty} \dot{r}(-\dot{r}) + \dot{r}(-\dot{r})\,dt = -2\int_{-\infty}^{\infty}\dot{r}^2\,dt = -\tfrac{4}{3},$$

$$b_{111} = \int_{-\infty}^{\infty} r(-6r^3) - r(-12r^3)\,dt = 6\int_{-\infty}^{\infty} r^4\,dt = 8,$$
$$b_{211} = \int_{-\infty}^{\infty} \dot{r}(-6r^3) + \dot{r}(-12r^3)\,dt = -18\int_{-\infty}^{\infty} \dot{r}r^3\,dt = 0,$$

$$M_1(\mu,\beta,\alpha) = [(A-B)\pi \operatorname{sech}\tfrac{\pi\omega}{2}\cos\omega\alpha]\mu_1 + 4\beta^2,$$
$$M_2(\mu,\beta,\alpha) = [(A-B)\pi\omega \operatorname{sech}\tfrac{\pi\omega}{2}\sin\omega\alpha]\mu_1 - \tfrac{4}{3}\mu_2,$$

$$D_{(\beta,\alpha)}M(\mu,\beta,\alpha) = \begin{bmatrix} [(B-A)\pi\omega \operatorname{sech}\tfrac{\pi\omega}{2}\sin\omega\alpha]\mu_1 & 8\beta \\ (B-A)\pi\omega^2 \operatorname{sech}\tfrac{\pi\omega}{2}\cos\omega\alpha]\mu_1 & 0 \end{bmatrix}.$$

Let us assume $A < B$, $A + B \neq 0$. The conditions for chaos in Theorem 3.2 are $M_1(\mu_0,\beta_0,\alpha_0) = M_2(\mu_0,\beta_0,\alpha_0) = 0$ and $\det\big(D_{(\beta,\alpha)}M(\mu_0,\beta_0,\alpha_0)\big) \neq 0$. These can be satisfied by taking $-\frac{\pi}{2} < \omega\alpha_0 < \frac{\pi}{2}$ and then

$$\frac{\mu_{0,2}}{\mu_{0,1}} = \tfrac{3\pi}{4}(A+B)\omega \operatorname{sech}\tfrac{\pi\omega}{2}\sin\omega\alpha_0,$$
$$\beta_0^2 = \tfrac{\pi}{4}(B-A)\mu_{0,1}\operatorname{sech}\tfrac{\pi\omega}{2}\cos\omega\alpha_0.$$

We again get a region as in Fig. 3.1.

In the case $A > B$, $A + B \neq 0$ we can take $\frac{\pi}{2} < \omega\alpha_0 < \frac{3\pi}{2}$. It is interesting to note that the transversality condition fails if $A + B = 0$.

Example 3.4. Consider the equations:

$$\ddot{x} = x - 2xz^2 + \dot{x}^2 + \mu_1\cos\omega t - \mu_2 z,$$
$$\ddot{y} = y - 2yz^2 + \dot{x}\dot{y},$$
$$\ddot{z} = z - 2z^3 + y\dot{y} + \mu_1\cos\omega t + (\mu_2 - \mu_1)\dot{z}.$$

We work in the phase space $(x,\dot{x},y,\dot{y},z,\dot{z})$. For the unperturbed system the eigenvalues of the origin are $\{-1,-1,-1,+1,+1,+1\}$ and a homoclinic solution is given by $x = y = 0$, $z = r$, i.e. $\gamma = (0,0,0,0,r,\dot{r})$.

Denoting a solution to the variational equation by $u = (\eta, \dot{\eta}, \zeta, \dot{\zeta}, \varphi, \dot{\varphi})$ we get

$$\ddot{\eta} = (1 - 2r^2)\eta, \qquad \ddot{\zeta} = (1 - 2r^2)\zeta, \qquad \ddot{\varphi} = (1 - 6r^2)\varphi$$

and so the variational equation has three bounded solutions given by

$$\begin{aligned} u_1 &= (r, \dot{r}, 0, 0, 0, 0), \\ u_2 &= (0, 0, r, \dot{r}, 0, 0), \\ u_3 &= \dot{\gamma} = (0, 0, 0, 0, \dot{r}, \ddot{r}). \end{aligned}$$

This shows that $d = 3$.

In a similar way we obtain three bounded solutions to the adjoint equations as

$$\begin{aligned} v_1 &= (-\dot{r}, r, 0, 0, 0, 0), \\ v_2 &= (0, 0, -\dot{r}, r, 0, 0), \\ v_3 &= (0, 0, 0, 0, -\ddot{r}, \dot{r}). \end{aligned}$$

Using these results, the equations $M = 0$ in Theorem 3.2 become:

$$\begin{aligned} M_1(\mu, \beta, \alpha) &= a_{11}\mu_1 + 2\mu_2 - \frac{\pi}{8}{\beta_1}^2 = 0, \\ M_2(\mu, \beta, \alpha) &= -\frac{\pi}{8}\beta_1\beta_2 = 0, \\ M_3(\mu, \beta, \alpha) &= a_{31}\mu_1 - \frac{2}{3}\mu_2 - \frac{\pi}{8}{\beta_2}^2 = 0 \end{aligned}$$

where

$$\begin{aligned} a_{11}(\alpha) &= -\pi \cos\omega\alpha \operatorname{sech} \tfrac{\pi\omega}{2}, \\ a_{31}(\alpha) &= \frac{2}{3} - \pi\omega \sin\omega\alpha \operatorname{sech} \frac{\pi\omega}{2}. \end{aligned}$$

We see that these equations yield two solutions, summarized below:

$$\text{(i)} \qquad \beta_0 = \begin{pmatrix} \sqrt{\frac{8}{\pi}(a_{11}(\alpha_0) + 3a_{31}(\alpha_0))} \\ 0 \end{pmatrix}, \qquad \mu_0 = \begin{pmatrix} 1 \\ \frac{3}{2}a_{31}(\alpha_0) \end{pmatrix},$$

$$\text{(ii)} \qquad \beta_0 = \begin{pmatrix} 0 \\ \sqrt{\frac{8}{3\pi}(a_{11}(\alpha)_0 + 3a_{31})} \end{pmatrix}, \qquad \mu_0 = \begin{pmatrix} 1 \\ -\frac{1}{2}a_{11}(\alpha_0) \end{pmatrix}.$$

For the respective cases we also have

(i) $\det(D_{\beta,\alpha}M(\mu_0, \beta_0, \alpha_0)) = -\frac{\pi^2\omega^2}{4}[a_{11}(\alpha_0) + 3a_{31}(\alpha_0)] \cos\omega\alpha_0 \operatorname{sech} \frac{\pi\omega}{2}$,

(ii) $\det(D_{\beta,\alpha}M(\mu_0, \beta_0, \alpha_0)) = \frac{\pi^2\omega}{12}[a_{11}(\alpha_0) + 3a_{31}(\alpha_0)] \sin\omega\alpha_0 \operatorname{sech} \frac{\pi\omega}{2}$.

Since $\lim_{\omega\to\infty}[a_{11}(\alpha) + 3a_{31}(\alpha)] = 2$ it follows that, for sufficiently large ω, $\det(D_{(\beta,\alpha)}M(\mu_0, \beta_0, \alpha_0)) \neq 0$ for all $\alpha_0 \in \mathbb{R}$ so we have two wedge-shaped regions in the μ_1-μ_2 plane within which the equation has a homoclinic solution. In the first region the limiting curves for the region have slopes given by

$$m_1 = 1 \pm \frac{3}{2}\pi\omega \operatorname{sech} \frac{\pi\omega}{2}.$$

For the second region the values are $m_2 = \pm\frac{\pi}{2} \operatorname{sech} \frac{\pi\omega}{2}$.

For large values of ω (rapid forcing) the asymptotic behavior is

$$m_1 \sim 1 \pm 3\pi\omega e^{-\frac{\pi\omega}{2}}, \quad m_2 \sim \pm\pi e^{-\frac{\pi\omega}{2}}.$$

Thus, the regions become very narrow as $\omega \to \infty$. This phenomenon has been demonstrated in other cases. See, for example, [8].

Example 3.5. This example uses rotational symmetry to obtain a homoclinic 2-manifold. Consider:

$$\ddot{x} = x - 2x(x^2 + y^2) + b_1\mu_1 \cos\omega_1 t + b_2\mu_2\dot{x},$$
$$\ddot{y} = y - 2y(x^2 + y^2) + b_3\mu_1 \cos\omega_2 t + b_4\mu_2\dot{y}$$

where b_i, ω_i are constants. Similar examples are given in [21] and [28]. When $\mu = 0$ the eigenvalues of the origin are $\{-1, -1, +1, +1\}$ and the system has a homoclinic solution given by $x(t) = r(t)\cos\beta$, $y(t) = r(t)\sin\beta$ for all β. We have a 2-manifold of homoclinic solutions so, necessarily, $d = 2$. We shall utilize Theorem 3.5.

The necessary calculations are simplified by introduction of the rotated coordinates $\bar{x} = x\cos\beta + y\sin\beta$, $\bar{y} = -x\sin\beta + y\cos\beta$. In the phase space $(\bar{x}, \dot{\bar{x}}, \bar{y}, \dot{\bar{y}})$ we have $\gamma_\beta = (r, \dot{r}, 0, 0)$ and $\dfrac{\partial\gamma}{\partial\beta} = (0, 0, r, \dot{r})$.

Solving the adjoint equation we get

$$v_{\beta 1} = (0, 0, -\dot{r}, r), \quad v_{\beta 2} = (-\ddot{r}, \dot{r}, 0, 0).$$

Using these results we get

$$a_{11}(\beta, \alpha) = \pi b_1 \sin\beta \cos\omega_1\alpha \operatorname{sech}\tfrac{\pi\omega_1}{2} - \pi b_3 \cos\beta \cos\omega_2\alpha \operatorname{sech}\tfrac{\pi\omega_2}{2},$$
$$a_{12}(\beta, \alpha) = 0,$$
$$a_{21}(\beta, \alpha) = -\pi b_1\omega_1 \cos\beta \sin\omega_1\alpha \operatorname{sech}\tfrac{\pi\omega_1}{2} - \pi b_3\omega_2 \sin\beta \sin\omega_2\alpha \operatorname{sech}\tfrac{\pi\omega_2}{2},$$
$$a_{22}(\beta, \alpha) = -\tfrac{2}{3}(b_2\cos^2\beta + b_4\sin^2\beta).$$

The bifurcation equations using (3.6) are

$$a_{11}(\beta, \alpha)\mu_1 = 0,$$
$$a_{21}(\beta, \alpha)\mu_1 + a_{21}(\beta, \alpha)\mu_2 = 0$$

Solution of the preceding equations requires $\det(A) = a_{11}a_{22} = 0$. We can satisfy this by taking

$$\beta_0 = \tan^{-1}\left[\frac{b_3\cos\omega_2\alpha_0 \operatorname{sech}\frac{\pi\omega_2}{2}}{b_1\cos\omega_1\alpha_0 \operatorname{sech}\frac{\pi\omega_1}{2}}\right], \qquad \mu_0 = (1, -a_{21}(\beta_0, \alpha_0)/a_{22}(\beta_0, \alpha_0)).$$

For fixed α_0 we get a bifurcation curve of slope $m = \mu_{0,2}$ when $D_{(\beta,\alpha)}M(\beta_0, \alpha_0)$ is nonsingular.

To make the results more explicit let us take each $b_i = 1$ and $\omega_1 = \omega_2 = \omega$. We can take $\beta_0 = \pi/4$ and

$$m_0 = -\tfrac{3\sqrt{2}}{2}\pi\omega \sin\omega\alpha_0 \operatorname{sech} t\frac{\pi\omega}{2}.$$

We also have $\beta_0 = 5\pi/4$ with the corresponding m the negative of that for $\beta_0 = \pi/4$. In the two cases we have

$$\det\big(D_{(\beta,\alpha)}M(\mu_0, \beta_0, \alpha_0)\big) = -\pi^2\omega^2\cos^2\omega\alpha_0 \operatorname{sech}^2\tfrac{\pi\omega}{2}.$$

We obtain a wedge-shaped region in the μ_1–μ_2 plane centered on the μ_1 axis bounded by curves of slopes

$$m_0 = \pm\frac{3\sqrt{2}}{2}\pi\omega \operatorname{sech}\frac{\pi\omega}{2}.$$

For each μ in this region the original differential equations have a transverse homoclinic solution from Theorem 3.5 and exhibit chaos. As before, the width of the region decreases exponentially as ω increases.

CHAPTER IV

EXTENSION TO CENTER MANIFOLDS AND INFINITE DIMENSIONS

In this chapter we discuss ongoing work toward extending our techniques to partial differential equations. The basic idea is to do a Bubnov-Galerkin-Kantorovich reduction to obtain an infinite system of equations with a finite hyperbolic part and an infinite center part. We project onto the hyperbolic part and apply the theory of Chapter III to obtain chaos and simultaneously introduce damping as a perturbation to make the center part weakly contracting. It then remains to show that the chaos from the hyperbolic part appears in the full equations. This work has been carried out for a finite dimensional center manifold [14]. Extension to the infinite dimensional case is under way.

A Finite Dimensional Center Manifold

To illustrate the ideas consider the equations

$$\ddot{x} = x - 2x(x^2 + y^2) - \mu_2\dot{x} + \mu_1 \cos\omega t, \tag{4.1a}$$

$$\ddot{y} = (1-k)y - 2y(x^2 + y^2) - \mu_2\dot{y} + \mu_1 \cos\omega t. \tag{4.1b}$$

This system consists of the radially symmetric Duffing oscillator of Example 3.5 with an additional spring of stiffness k in the y equation along with the same damping and external forces added as perturbation terms.

Let us assume $k > 1$ in (4.1b). Then, for the unperturbed equation i.e., when $\mu_1 = \mu_2 = 0$, the linear part of (4.1a) has a hyperbolic equilibrium and the linear part of (4.1b) has a center. Furthermore, for small μ_2, the eigenvalues of $\ddot{y} = (1-k)y - \mu_2\dot{y}$ are complex functions, $\lambda(\mu_2)$, with $\Re(\lambda(\mu_2)) = -\mu_2/2$ so that we have $\Re(\lambda(0)) = 0$ and $\Re(\lambda'(0)) = -1/2$. Thus, for small $\mu_2 > 0$, the equilibrium of (4.1b) is weakly asymptotically stable.

If we set $y = 0$ in (4.1a) we get the standard forced, damped Duffing equation. From Example 3.1 we know that for small $\mu_1 \neq 0$ and for $\mu_2 \neq 0$, within an appropriate range, this equation has a transverse homoclinic orbit and hence exhibits chaos. The idea now is to show that if $\mu_1 \neq 0$, $\mu_2 > 0$ are chosen to produce chaos in (4.1a) when $y = 0$ and if $|\mu_2/\mu_1|$ is sufficiently large then, as a consequence of the weak contraction in the y equation, there exists chaos in the full equation (4.1) which, in some sense, shadows the chaos obtained in (4.1a) with $y = 0$.

As an abstract version of (4.1) we consider differential equations of the form

$$\begin{aligned}\dot{x} &= f(x, y, \mu, t)\\ &= f_0(x, y) + \mu_1 f_1(x, y, \mu, t) + \mu_2 f_2(x, y, \mu, t),\end{aligned} \tag{4.2a}$$

$$\begin{aligned}\dot{y} &= g(x, y, \mu, t)\\ &= g_0(x, y) + \mu_1 g_1(x, y, \mu, t) + \mu_2 g_2(x, y, \mu)\end{aligned} \tag{4.2b}$$

with $x \in \mathbb{R}^{n_1}$, $y \in \mathbb{R}^{n_2}$, $\mu = (\mu_1, \mu_2) \in \mathbb{R}^2$.

We make the following assumptions about (4.2):

(i) Each f_i, g_i is $\mathcal{C}^3$ in all arguments.

(ii) f_1, f_2 and g_1 are periodic in t with period T.

(iii) $f_0(0,0) = 0$ and $D_2 f_0(x,0) = 0$.

(iv) The eigenvalues of $D_1 f_0(0,0)$ lie off the imaginary axis.

(v) The equation $\dot{x} = f_0(x,0)$ has a nontrivial homoclinic solution γ.

(vi) $g_0(x,0) = g_2(x,0,\mu) = 0$.

(vii) The eigenvalues of $D_2 g_0(0,0)$ lie on the imaginary axis.

(viii) If $\mu_2 \to \lambda(\mu_2)$ is a function such that $\lambda(\mu_2)$ is an eigenvalue of the matrix $D_2 g_0(0,0) + \mu_2 D_2 g_2(0,0,0)$ then $\Re(\lambda'(0)) < 0$.

Chaotic Dynamics on the Hyperbolic Subspace

Consider the equation

$$\dot{x} = f(x,0,\mu,t) = f_0(x,0) + \mu_1 f_1(x,0,\mu,t) + \mu_2 f_2(x,0,\mu,t) \tag{4.3}$$

obtained by setting $y = 0$ in (4.2a). Equation (4.3) will be called the reduced equation obtained from (4.2). The following result is due to Palmer [26] with a slight extension for our purposes below. Part (d) is Theorem 2.6 and establishes chaos for (4.3). The additional details will be used in extending the results to the full equation (4.2).

4.1 THEOREM.

a) *Let μ_0, α_0, β_0, ξ_0, γ_ξ be obtained by applying Theorem 3.2 or Theorem 3.5 to (4.3). Fix $\xi \in (0, \xi_0]$ and let N be a positive integer. Then there exist an $\epsilon_0 > 0$ and a function $\epsilon \to M_N(\epsilon) > N$ such that given ϵ with $0 < \epsilon \le \epsilon_0$ and a positive integer $m \ge M_N(\epsilon)$ the equation $\dot{x} = f(x,0,\xi\mu_0,t)$ has for each $\sigma \in \Sigma_N$ a unique solution $t \to x_\sigma(t)$ satisfying*

$$|x_\sigma(t + (2k-1)mT) - \gamma_\xi(t + \sigma_k T)| \le \epsilon \quad \text{for } -mT \le t \le mT,\ k \in \mathbb{N}.$$

b) *If σ, $\sigma' \in \Sigma_N$, and $\sigma_k \ne \sigma'_k$ then*

$$|x_\sigma((2k-1)mT - \sigma_k T) - x_{\sigma'}((2k-1)mT - \sigma'_k T)| > \epsilon.$$

c) *x_σ depends continuously on σ and $x_\sigma(t + 2mT) = x_{\varphi(\sigma)}(t)$ where φ is the Bernoulli shift on Σ_N.*

d) *The correspondence $\phi(\sigma) = x_\sigma(0)$ is a homeomorphism of Σ_N onto a compact subset of $\mathbb{R}^{n_1}$ on which the $2m$th iterate F^{2m} of the period map F is invariant and satisfies $F^{2m} \circ \phi = \phi \circ \varphi$.*

e) *The variational equation $\dot{u} = D_1 f_0(x_\sigma(t), 0) u$ has an exponential dichotomy on $[0, \infty)$ with constants independent of σ and ϵ.*

f) *Given $\mu_{0,2} > 0$ there exist constants $b > 0$, $B > 0$ independent of σ such that the variational equation*

$$\dot{v} = [D_2 g_0(x_\sigma(t), 0) + \xi\mu_{0,2} D_2 g_2(x_\sigma(t), 0, 0)]\, v$$

has a fundamental solution, V, on $[0,\infty)$ satisfying

$$\left|V(t)V(s)^{-1}\right| \le Be^{\xi b\mu_{0,2}(s-t)} \quad \textit{for all } 0 \le s \le t.$$

PROOF. Parts (a)-(d) follow from Corollary 3.6 of [26] and its proof. To show (e) consider the variational equation

$$\dot{u} = D_1 f_0(x_\sigma(t),0)u. \tag{4.4}$$

This takes the form

$$\dot{u} = D_1 f_0(\gamma_\xi(t-(2k-1)mT+\sigma_k T),0)u + B_k(t)u, \quad t_{k-1} \le t \le t_k \tag{4.5}$$

where $t_k = 2kmT$ and

$$B_k(t) = D_1 f_0(x_\sigma(t),0) - D_1 f_0(\gamma_\xi(t-(2k-1)mT+\sigma_k T),0,).$$

Let (U,P) be an exponential dichotomy on $[0,\infty)$ with constants (A,a) for the equation $\dot{u} = D_1 f(\gamma_\xi(t),0)u$. This exists by Theorem 2.2 since $D_1 f(\gamma_\xi(t),0) \to D_1 f_0(0,0)$ at an exponential rate as $t\to\infty$. Then the equation

$$\dot{u} = D_1 f_0(\gamma_\xi(t-(2k-1)mT+\sigma_k T),0)u, \quad t \ge (2k-1)mT - \sigma_k T$$

has an exponential dichotomy (U_k, P) with the same constants where

$$U_k(t) = U(t-(2k-1)mT+\sigma_k T) = U(t-t_{k-1}+\sigma_k T).$$

Let $\delta > 0$ be given. By the proof of Theorem 3.5 in [26] we can assume ϵ_0 and $M_N(\epsilon)$ are chosen so that

$$|U_{k-1}(t_{k-1})PU_{k-1}(t_{k-1})^{-1} - U_k(t_{k-1})PU_k(t_{k-1})^{-1}| < \delta \quad \text{for } t_{k-1} \le t \le t_k \tag{4.6}$$

and

$$|B_k(t)| < \delta \qquad \text{for } t_{k-1} \le t \le t_k.$$

Since δ was arbitrary it follows from the roughness theorem [6, p. 34] that (4.5) has an exponential dichotomy $(\tilde{U}_k, Q)$ with constants $((5/2)A^2, a-2A\delta)$ such that

$$|\tilde{U}_k(t)Q\tilde{U}_k(t)^{-1} - U_k(t)PU_k(t)| \le 4a^{-1}A^3\delta \qquad \text{for } t_{k-1} \le t \le t_k. \tag{4.7}$$

Combining (4.6) and (4.7) we get

$$|\tilde{U}_{k-1}(t_{k-1})Q\tilde{U}_{k-1}(t_{k-1})^{-1} - \tilde{U}_k(t_{k-1})Q\tilde{U}_k(t_{k-1})^{-1}| \le (8a^{-1}A^3+1)\delta.$$

We now obtain an exponential dichotomy on $[0,\infty)$ for (4.4) from Lemma 3.2 of [26]. This proves part (e).

Part (f) follows in a similar way. First, each eigenvalue, $\lambda(\xi)$, for the matrix $D_2 g_0(0,0) + \xi\mu_{0,2} D_2 g_2(0,0,0)$ satisfies $\Re(\lambda(0)) = 0$ and $\Re(\lambda'(0)) < 0$ by hypotheses

(vii) and (viii) for (4.2). Thus, we can assume ξ_0 is chosen small enough so that each eigenvalue satisfies $\Re(\lambda(\xi)) \leq -4b\xi\mu_{0,2}$ for some $b > 0$.

Since $\gamma_\xi(t) \to 0$ exponentially as $t \to \infty$, we have

$$D_2g_0(\gamma_\xi(t),0) + \xi\mu_{0,2}D_2g_2(\gamma_\xi(t),0,0) \to D_2g_0(0,0) + \xi\mu_{0,2}D_2g_2(0,0,0) \quad \text{as } t \to \infty$$

also exponentially. By Theorem 2.2 the equation

$$\dot{v} = [D_2g_0(\gamma_\xi(t),0) + \xi\mu_{0,2}D_2g_2(\gamma_\xi(t),0,0)]v$$

has a fundamental solution, V_0, satisfying $|V_0(t)V_0(s)| \leq B_0e^{3b\xi\mu_{0,2}(s-t)}$ for all $0 \leq s \leq t$ for some $B_0 > 0$.

We can assume ϵ_0 and $M_N(\epsilon)$ are chosen so that

$$\big|D_2g_0(x_\sigma(t),0) + \xi\mu_{0,2}D_2g_2(x_\sigma(t),0,0) - D_2g_0(\gamma_\xi(t-(2k-1)mT+\sigma_kT),0) - \xi\mu_{0,2}D_2g_2(\gamma_\xi(t-(2k-1)mT+\sigma_kT),0,0)\big| < \delta$$

for $t_{k-1} \leq t \leq t_k$ where δ is arbitrary.

Hence, by the roughness theorem, there exists a fundamental solution, V_k, for the equation

$$\dot{v} = [D_2g_0(x_\sigma(t),0) + \xi\mu_{0,2}D_2g_2(x_\sigma(t),0,0)]v, \quad t_{k-1} \leq t \leq t_k$$

with $|V_k(t)V_k(s)^{-1}| \leq Be^{2b\xi\mu_{0,2}(s-t)}$ for $t_{k-1} \leq s \leq t \leq t_k$. We now obtain a fundamental solution on $[0,\infty)$ from Lemma 3.1 of [26]. This proves part (f). □

Chaos in the Full Equation

In the preceding section we established chaos for the reduced equation (4.3). We now establish a condition for chaos to exist in the full equation. We do this by shadowing the functions x_σ in Theorem 4.1.

Let μ_0, α_0, β_0, ξ_0, γ_ξ be obtained by applying Theorem 3.2 or Theorem 3.5 to (4.3). Fix $\xi \in (0,\xi_0]$ and an integer $N \geq 0$. We now turn to Theorem 4.1 where we first obtain ϵ_0 and M_N. Then, we fix $\epsilon \in (0,\epsilon_0]$, $m \geq M_N(\epsilon)$, $\sigma \in \Sigma_N$ and obtain x_σ. In (4.2) we substitute $\mu = \xi\mu_0$ and make the change of variable $x = x_\sigma + w$. The equations for w and y can be written

$$\dot{w} = D_1f_0(x_\sigma,0)w + h_1(w,y,\sigma,\xi,t), \tag{4.8a}$$

$$\dot{y} = [D_2g_0(x_\sigma,0) + \xi\mu_{0,2}D_2g_2(x_\sigma,0,0)]\,y + h_2(w,y,\sigma,\xi,t) \tag{4.8b}$$

where

$$\begin{aligned} h_1(w,y,\sigma,\xi,t) = {} & f_0(x_\sigma(t)+w,y) - f_0(x_\sigma(t),0) - D_1f_0(x_\sigma(t),0)w \\ & + \xi\mu_{0,1}\left[f_1(x_\sigma(t)+w,y,\xi\mu_0,t) - f_1(x_\sigma(t),0,\xi\mu_0,t)\right] \\ & + \xi\mu_{0,2}\left[f_2(x_\sigma(t)+w,y,\xi\mu_0,t) - f_2(x_\sigma(t),0,\xi\mu_0,t)\right], \end{aligned}$$

$$\begin{aligned} h_2(w,y,\sigma,\xi,t) = {} & g_0(x_\sigma(t)+w,y) - D_2g_0(x_\sigma(t),0)y \\ & + \xi\mu_{0,1}g_1(x_\sigma(t)+w,y,\xi\mu_0,t) \\ & + \xi\mu_{0,2}\left[g_2(x_\sigma(t)+w,y,\xi\mu_0) - D_2g_2(x_\sigma(t),0,0)y\right]. \end{aligned}$$

4.2 THEOREM. *There exist positive constants $\bar{\xi}_0$, ϵ_0 independent of σ, so that given $\xi \in (0, \bar{\xi}_0]$, and $\epsilon \in (0, \epsilon_0]$ there exists a constant $C > 0$ such that if $\mu_{0,2} > 0$ and $|\mu_{0,2}/\mu_{0,1}| > C$ then there exists a unique solution, (w_σ, y_σ), of (4.8) satisfying $|w_\sigma(t)| + |y_\sigma(t)| \le \frac{1}{4}\epsilon$ for $t \ge 0$ Furthermore, (w_σ, y_σ) depends continuously on σ and $\big(w_\sigma(t + 2mT), y_\sigma(t + 2mT)\big) = \big(w_{\varphi(\sigma)}(t), y_{\varphi(\sigma)}(t)\big)$ where the integer m is as in Theorem 4.1a and φ is the Bernoulli shift on Σ_N.*

PROOF. Let $r = 2\sup_t |\gamma(t)|$ and choose $K > 0$ so that the following are satisfied:

$$\begin{aligned}
&\sup_{|x|\le r}\sup_{|\mu|\le 1} |g_1(x,0,\mu,t)| \le K;\\
&\sup_{|x|\le r}\sup_{|\mu|\le 1} |D_i f_j(x,0,\mu,t)| \le K, \qquad i,j = 1,2;\\
&\sup_{|x|\le r}\sup_{|\mu|\le 1} |D_i g_1(x,0,\mu,t)| \le K, \qquad i = 1,2;\\
&\sup_{|x|\le r}\sup_{|\mu|\le 1} |D_2 g_2(x,0,\mu) - D_2 g_2(x,0,0)| \le K|\mu|.
\end{aligned}$$

By rescaling ξ we can assume $|\mu_0| = 1$ and by choosing ξ_0 sufficiently small in Theorem 3.2 or Theorem 3.5 and ϵ_0 sufficiently small in Theorem 4.1a we can assume that $\xi_0 \le 1$ and that $|x_\sigma(t)| \le r$ for all t and all $\sigma \in \Sigma_N$.

Define the Banach space $\mathbb{X}$ by

$$\mathbb{X} = \left\{ \eta \in \mathcal{C}^0\left(\mathbb{R}^+, \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}\right) \,\Big|\, \sup_{t\ge 0}(|\eta_1(t)| + |\eta_2(t)|) < \infty \right\}$$

with norm the supremum in the definition. Let ξ_0 be as in Theorem 3.2 or Theorem 3.5, let (U, P) be the exponential dichotomy, with constants (A, a), obtained from Theorem 4.1(e) and, for $\xi \in (0, \xi_0]$, let (V, I) be the exponential dichotomy, with constants $(B, \xi b \mu_{0,2})$, obtained from Theorem 4.1(f). Now consider the function $F : \mathbb{X} \times \Sigma_N \to \mathbb{X}$ defined by $F = (F_1, F_2)$ where

$$\begin{aligned}
F_1(\eta,\sigma)(t) = U(t) &\int_0^t PU(s)^{-1} h_1(\eta_1(s), \eta_2(s), \sigma, \xi, s)\, ds\\
&- U(t) \int_t^\infty (I - P)U(s)^{-1} h_1(\eta_1(s), \eta_2(s), \sigma, \xi, s)\, ds,\\
F_2(\eta,\sigma)(t) = V(t) &\int_0^t V(s)^{-1} h_2(\eta_1(s), \eta_2(s), \sigma, \xi, s)\, ds.
\end{aligned}$$

F is differentiable in η and, from Theorem 4.1(c), F is continuous in σ. Fixed points of F are bounded solutions to (4.8).

Note that from the definition of K along with (iii) and (vi) for (4.2) we have

$$|D_i h_1(0,0,\sigma,\xi,t)| \le 2K\xi, \qquad i = 1,2 \tag{4.9a}$$

$$|D_1 h_2(0,0,\sigma,\xi,t)| \le K\xi|\mu_{0,1}|, \tag{4.9b}$$

$$|D_2 h_2(0,0,\sigma,\xi,t)| \le K\xi|\mu_{0,1}| + K\xi^2\mu_{0,2}. \tag{4.9c}$$

These results yield

$$|D_{\eta_1} F_1(0,\sigma)(t)| \le \int_{-\infty}^{t} Ae^{a(s-t)} 2K\xi \, ds + \int_{t}^{\infty} Ae^{a(t-s)} 2K\xi \, ds = \frac{4K\xi A}{a}.$$

In a similar way we get $|D_{\eta_2} F_1(0,\sigma)(t)| \le (4K\xi A)/a$. Thus, we can choose $\xi_1 > 0$ so that $\|D_\eta F_1(0,\sigma)\| \le 1/16\sqrt{2}$ if $0 < \xi \le \xi_1$.

Next we compute

$$|D_{\eta_1} F_2(0,\sigma)(t)| \le \int_0^t Be^{\xi b \mu_{0,2}(s-t)} K\xi |\mu_{0,1}| \, ds \le \frac{BK}{b} \left| \frac{\mu_{0,1}}{\mu_{0,2}} \right|,$$

$$\begin{aligned} |D_{\eta_2} F_2(0,\sigma)(t)| &\le \int_0^t Be^{\xi b \mu_{0,2}(s-t)} \left(K\xi |\mu_{0,1}| + K\xi^2 \mu_{0,2} \right) ds \\ &\le \frac{BK}{b} \left(\left| \frac{\mu_{0,1}}{\mu_{0,2}} \right| + \xi \right). \end{aligned}$$

Using these results we can choose $\delta_1 > 0$, $\xi_2 > 0$ so that

$$\|D_\eta F_2(0,\sigma)\| \le \frac{1}{16\sqrt{2}} \qquad \text{if } 0 < \xi \le \xi_2, \quad 0 < \left| \frac{\mu_{0,1}}{\mu_{0,2}} \right| \le \delta_1.$$

Let $\bar{\xi}_0 = \min\{\xi_0, \xi_1, \xi_2\}$. Then

$$\|D_\eta F(0,\sigma)\| \le \frac{1}{16} \qquad \text{if } 0 < \xi \le \bar{\xi}_0,\ 0 \le \left| \frac{\mu_{0,1}}{\mu_{0,2}} \right| \le \delta_1.$$

From this last result we can choose $\epsilon_0 > 0$ so that, with the same restrictions, $\|D_\eta F(\eta,\sigma)\| \le \frac{1}{8}$ if $\|\eta\| \le \epsilon_0$.

Using (vi) for (4.2) and the definition of K we have

$$\begin{aligned} &h_1(0,0,\sigma,\xi,t) = 0, \\ &|h_2(0,0,\sigma,\xi,t)| \le K\xi |\mu_{0,1}| \end{aligned}$$

so that $F_1(0,\sigma) = 0$ and

$$|F_2(0,\sigma)(t)| \le \int_0^t Be^{\xi b \mu_{0,2}(s-t)} K\xi |\mu_{0,1}| \, ds \le \frac{BK}{b} \left| \frac{\mu_{0,1}}{\mu_{0,2}} \right|.$$

Given $\epsilon \in (0,\epsilon_0]$ we can, using the preceding formula, choose $\delta_2 > 0$ so that $\|F(0,\sigma)\| \le \frac{1}{8}\epsilon$ if $|\mu_{0,1}/\mu_{0,2}| \le \delta_2$. With $\delta = \min\{\delta_1,\delta_2\}$ and $C = 1/\delta$, F is a uniform contraction on the ball of radius $\frac{1}{4}\epsilon$ with center at the origin. The existence now follows from the uniform contraction mapping theorem. See Theorem 3.2 of Hale [16].

We now show uniqueness. Suppose (w_1, y_1) and (w_2, y_2) are two solutions to (4.8) satisfying $|w_i(t)| + |y_i(t)| \le \epsilon$ for $t \ge 0$. Using (4.9) we can assume ϵ_0, $\bar{\xi}_0$, δ

have been chosen sufficiently small so that when $|w| \le \epsilon_0$, $|y| \le \epsilon_0$, $0 < \xi \le \bar{\xi}_0$, $|\mu_{0,1}/\mu_{0,2}| < \delta$ the following hold for $i = 1, 2$:

$$|D_i h_1(w, y, \sigma, \xi, t)| \le \frac{a}{8A},$$
$$|D_i h_2(w, y, \sigma, \xi, t)| \le \frac{\xi b \mu_{0,2}}{8B}.$$

We have

$$\dot{w}_1(t) - \dot{w}_2(t) = D_1 f_0(x_\sigma(t), 0)\,(w_1(t) - w_2(t)) + \bar{h}(t)$$

where

$$\bar{h}(t) = h_1(w_1(t), y_1(t), \sigma, \xi, t) - h_1(w_2(t), y_2(t), \sigma, \xi, t)$$

so that

$$w_1(t) - w_2(t) = U(t)\int_0^t PU(s)^{-1}\bar{h}(s)\,ds - U(t)\int_t^\infty (I - P)U(s)^{-1}\bar{h}(s)\,ds.$$

Let $s_1 = \sup_{t \ge 0}|w_1(t) - w_2(t)|$, $s_2 = \sup_{t \ge 0}|y_1(t) - y_2(t)|$. Using Taylor's formula and taking norms we get

$$s_1 \le \int_0^t Ae^{a(s-t)}\frac{a}{8A}(s_1 + s_2)\,ds + \int_t^\infty Ae^{a(t-s)}\frac{a}{8A}(s_1 + s_2)\,ds \le \frac{1}{4}(s_1 + s_2).$$

In a similar way we get $s_2 \le \frac{1}{4}(s_1 + s_2)$. From this we conclude $s_1 + s_2 \le \frac{1}{2}(s_1 + s_2)$ so that, since each $s_i \ge 0$, $s_1 = s_2 = 0$. Hence, for each σ we have a unique (w_σ, y_σ).

Using $x_\sigma(t+2mT) = x_{\varphi(\sigma)}(t)$, from Theorem 4.1 and the periodicity in t of f_i, g_i it is easy to check that the function $t \to \big(w_\sigma(t+2mT), y_\sigma(t+2mT)\big)$ is a solution to (4.8) with σ replaced with $\varphi(\sigma)$. Thus, by uniqueness, $\big(w_\sigma(t+2mT), y_\sigma(t+2mT)\big) = \big(w_{\varphi(\sigma)}(t), y_{\varphi(\sigma)}(t)\big)$. □

We now come to the main result of this chapter which uses the solutions $(x_\sigma + w_\sigma, y_\sigma)$ for (4.2) to obtain chaos. The following proof follows Palmer [26].

4.3 THEOREM. *Let M be as in (3.5) or (3.6) and suppose $(\mu_0, \alpha_0, \beta_0)$ are such that $M(\mu_0, \alpha_0, \beta_0) = 0$ and $D_{(\alpha,\beta)}M(\mu_0, \alpha_0, \beta_0)$ is nonsingular. Then there exist $\bar{\xi}_0 > 0$, $C > 0$ such that if $0 < \xi \le \bar{\xi}_0$, if the parameters in (4.2) are given by $\mu = \xi\mu_0$, if $|\mu_{0,2}/\mu_{0,1}| > C$ and if $\mu_{0,2} > 0$ then there exists a homeomorphism, ϕ, of Σ_N onto a compact subset of $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ on which the $2m$th iterate, F^{2m}, of the period map F is invariant and satisfies $F^{2m} \circ \phi = \phi \circ \varphi$ where φ is the Bernoulli shift on Σ_N.*

PROOF. Suppose $M(\mu_0, \alpha_0, \beta_0) = 0$ and $D_{(\alpha,\beta)}M(\mu_0, \alpha_0, \beta_0)$ is nonsingular. Take the minimum ϵ_0 from Theorems 4.1 and 4.2, then fix $\epsilon \in (0, \epsilon_0]$ and $m \ge M_N(\epsilon)$ where M_N is from Theorem 4.1a. For each $\sigma \in \Sigma_N$ we obtain a unique x_σ from Theorem 4.1a.

Now obtain b from Theorem 4.1(f) and $\bar{\xi}_0$, C, (w_σ, y_σ) all from Theorem 4.2. Note that $|w_\sigma(t)| \le \frac{1}{4}\epsilon$ for $t \ge 0$ and all $\sigma \in \Sigma_N$.

We now define $\phi : \Sigma_N \to \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ by $\phi(\sigma) = (x_\sigma(0) + w_\sigma(0), y_\sigma(0))$. This function is continuous by the continuity of x_σ, w_σ, y_σ in σ. Suppose $\sigma, \sigma' \in \Sigma_N$

with $\sigma \neq \sigma'$. Then $\sigma_k \neq \sigma'_k$ for some k. Choose an integer p such that $\bar{t} \geq 0$ where $\bar{t} = (2(k+p)-1)mT - \sigma_k T$.

We now have, using Theorem 4.1b,

$$\begin{aligned}
\epsilon &\leq |x_{\varphi^p(\sigma)}(\bar{t}) - x_{\varphi^p(\sigma')}(\bar{t})| \\
&\leq |x_{\varphi^p(\sigma)}(\bar{t}) - (x_{\varphi^p(\sigma)}(\bar{t}) + w_{\varphi^p(\sigma)}(\bar{t}))| \\
&\quad + |(x_{\varphi^p(\sigma)}(\bar{t}) + w_{\varphi^p(\sigma)}(\bar{t})) - (x_{\varphi^p(\sigma')}(\bar{t}) + w_{\varphi^p(\sigma')}(\bar{t}))| \\
&\quad + |(x_{\varphi^p(\sigma')}(\bar{t}) + w_{\varphi^p(\sigma')}(\bar{t})) - x_{\varphi^p(\sigma')}(\bar{t})| \\
&\leq \frac{1}{4}\epsilon + |(x_{\varphi^p(\sigma)}(\bar{t}) + w_{\varphi^p(\sigma)}(\bar{t})) - (x_{\varphi^p(\sigma')}(\bar{t}) + w_{\varphi^p(\sigma')}(\bar{t}))| + \frac{1}{4}\epsilon.
\end{aligned}$$

Thus, $|(x_{\varphi^p(\sigma)}(\bar{t}) + w_{\varphi^p(\sigma)}(\bar{t})) - (x_{\varphi^p(\sigma')}(\bar{t}) + w_{\varphi^p(\sigma')}(\bar{t}))| \geq \epsilon/2$. It follows that the two solutions

$$\begin{aligned}
t &\to (x_{\varphi^p(\sigma)}(t) + w_{\varphi^p(\sigma)}(t), y_{\varphi^p(\sigma)}(t)), \\
t &\to (x_{\varphi^p(\sigma')}(t) + w_{\varphi^p(\sigma')}(t), y_{\varphi^p(\sigma')}(t))
\end{aligned}$$

are distinct and hence $\phi(\varphi^p(\sigma)) \neq \phi(\varphi^p(\sigma'))$ so ϕ is one-to-one. Since ϕ is continuous, one-to-one and Σ_N is compact, ϕ is a homeomorphism onto its compact image.

We also have, using Theorem 4.1(c) and Theorem 4.2,

$$\begin{aligned}
F^{2m}(\phi(\sigma)) &= F^{2m}(x_\sigma(0) + w_\sigma(0), y_\sigma(0)) \\
&= (x_\sigma(2mT) + w_\sigma(2mT), y_\sigma(2mT)) \\
&= (x_{\varphi(\sigma)}(0) + w_{\varphi(\sigma)}(0), y_{\varphi(\sigma)}(0)) \\
&= \phi(\varphi(\sigma)).
\end{aligned}$$

□

Some Finite Dimensional Examples

We now illustrate the above theory with two examples. As before let us denote $r(t) = \operatorname{sech} t$ and recall that $\ddot{r} = r - 2r^3$ and $\dddot{r} = (1-6r^2)\dot{r}$.

Example 4.1. As our first example consider the equations from the beginning of the chapter which we repeat here

$$\begin{aligned}
\ddot{x} &= x - 2x(x^2+y^2) - \mu_2\dot{x} + \mu_1 \cos\omega t, \\
\ddot{y} &= (1-k)y - 2y(x^2+y^2) - \mu_2\dot{y} + \mu_1 \cos\omega t.
\end{aligned}$$

The reduced equation is

$$\ddot{x} = x - 2x^3 - \mu_2\dot{x} + \mu_1 \cos\omega t.$$

From Example 3.1 we can take μ_0 to satisfy $\left|\frac{\mu_{0,2}}{\mu_{0,1}}\right| \leq m_0 = \frac{3\pi\omega}{4} \operatorname{sech} \frac{\pi\omega}{2}$.

Now, from Theorem 4.3, there exists a constant $C > 0$ such that the full equation exhibits chaos for all sufficiently small $|\mu_0|$ satisfying $\mu_{0,2} > 0$ and $C < |\mu_{0,2}/\mu_{0,1}| < m_0$. This is illustrated in Figure 4.1

FIGURE 4.1. Chaotic Region for Example 4.1

Example 4.2. As a generalization of the preceding example consider the equations

$$\begin{aligned}
\ddot{x} &= x - 2x(x^2 + y^2 + z^2) - \mu_2(\dot{x} + \dot{y}) + \mu_1 \cos \omega t, \\
\ddot{y} &= y - 2y(x^2 + y^2 + z^2) - \mu_2(\dot{x} + \dot{y}), \\
\ddot{z} &= (1 - k)z - 2z(x^2 + y^2 + z^2) - \mu_2 \dot{z} + \mu_1 \cos \omega t
\end{aligned}$$

where, as before, we assume $k > 1$. We consider these equations as a first order system in the phase space $(x, \dot{x}, y, \dot{y}, z, \dot{z})$.

The reduced equations are

$$\begin{aligned}
\ddot{x} &= x - 2x(x^2 + y^2) - \mu_2(\dot{x} + \dot{y}) + \mu_1 \cos \omega t, \\
\ddot{y} &= y - 2y(x^2 + y^2) - \mu_2(\dot{x} + \dot{y}).
\end{aligned}$$

The unperturbed potion of this system has a homoclinic 2-manifold with a family of homoclinic orbits given by $x = r(t) \cos \beta$, $y = r(t) \sin \beta$. Writing out the adjoint equation in $\mathbb{R}^4$ we obtain as a basis for the space of bounded solutions

$$\begin{aligned}
v_{\beta 1} &= (-\ddot{r} \cos \beta, \dot{r} \cos \beta, -\ddot{r} \sin \beta, \dot{r} \sin \beta), \\
v_{\beta 2} &= (-\dot{r} \sin \beta, r \sin \beta, \dot{r} \cos \beta, -r \cos \beta).
\end{aligned}$$

Next we compute

$$a_{11}(\alpha,\beta) = \int_{-\infty}^{\infty} \dot{r}\cos\beta\cos\omega(t+\alpha)\,dt = \pi\omega\,\mathrm{sech}\,\tfrac{\pi\omega}{2}\sin\omega\alpha\cos\beta,$$

$$\begin{aligned} a_{12}(\alpha,\beta) &= \int_{-\infty}^{\infty} -\dot{r}\cos\beta(\dot{r}\cos\beta+\dot{r}\sin\beta) - \dot{r}\sin\beta(\dot{r}\cos\beta+\dot{r}\sin\beta)\,dt \\ &= -\tfrac{2}{3}(\cos\beta+\sin\beta)^2, \end{aligned}$$

$$a_{21}(\alpha,\beta) = \int_{-\infty}^{\infty} r\sin\beta\cos\omega(t+\alpha)\,dt = \pi\,\mathrm{sech}\,\tfrac{\pi\omega}{2}\cos\omega\alpha\sin\beta,$$

$$a_{22}(\alpha,\beta) = \int_{-\infty}^{\infty} r\sin\beta(\dot{r}\cos\beta+\dot{r}\sin\beta) + r\cos\beta(\dot{r}\cos\beta+\dot{r}\sin\beta)\,dt = 0.$$

In (3.6), $d = 2$ and β is a scalar. The bifurcation equations take the form

$$\begin{aligned} a_{11}(\alpha,\beta)\mu_1 + a_{12}(\alpha,\beta)\mu_2 &= 0, \\ a_{21}(\alpha,\beta)\mu_1 \qquad\qquad\qquad &= 0. \end{aligned}$$

A sufficient condition for a nontrivial solution is $a_{21} = 0$ which is satisfied by taking $\omega\alpha_0 = \pi/2$. We then have

$$\frac{\mu_{0,2}}{\mu_{0,1}} = -\frac{a_{11}(\alpha_0,\beta_0)}{a_{12}(\alpha_0,\beta_0)} = \frac{3\pi\omega\,\mathrm{sech}\,\frac{\pi\omega}{2}\cos\beta_0}{2(\cos\beta_0+\sin\beta_0)^2}.$$

It is easy to see that by varying the parameter β_0 we obtain bifurcation curves through the origin in the μ_1-μ_2 plane of all slopes.

It remains to check the transversality condition which takes the form

$$\begin{aligned} &\det\left(D_{(\alpha,\beta)}M(\alpha_0,\beta_0,\mu_0)\right) \\ &= \begin{vmatrix} \frac{\partial a_{11}}{\partial\alpha}(\alpha_0,\beta_0)\mu_{0,1} + \frac{\partial a_{12}}{\partial\alpha}(\alpha_0,\beta_0)\mu_{0,2} & \frac{\partial a_{11}}{\partial\beta}(\alpha_0,\beta_0)\mu_{0,1} + \frac{\partial a_{12}}{\partial\beta}(\alpha_0,\beta_0)\mu_{0,2} \\ \frac{\partial a_{21}}{\partial\alpha}(\alpha_0,\beta_0)\mu_{0,1} + \frac{\partial a_{22}}{\partial\alpha}(\alpha_0,\beta_0)\mu_{0,2} & \frac{\partial a_{21}}{\partial\beta}(\alpha_0,\beta_0)\mu_{0,1} + \frac{\partial a_{22}}{\partial\beta}(\alpha_0,\beta_0)\mu_{0,2} \end{vmatrix} \\ &= -\frac{(\mu_{0,1})^2\pi^2\omega^2(\sin\beta_0 + 2\cos^3\beta_0)\sin\beta_0\,\mathrm{sech}^2\,\frac{\pi\omega}{2}}{(\cos\beta_0+\sin\beta_0)^2} \neq 0 \end{aligned}$$

We see that this condition is satisfied for β_0 in the set

$$\left\{\beta\in[0,2\pi] \mid \beta\notin\{0,\tfrac{3\pi}{4},\pi,\tfrac{7\pi}{4},2\pi\}\right\}.$$

Thus, the reduced equation exhibits chaos for all sufficiently small $|\mu_0|$ in the μ_1-μ_2 plane except along the lines of slope $m = \pm m_0$ where $m_0 = (3\pi\omega/2)\,\mathrm{sech}(\pi\omega/2)$. Then, from Theorem 4.3, there exists a constant $C > 0$ such that the full equation exhibits chaos for all sufficiently small $|\mu_0|$ lying above the two lines with slope $m = \pm C$ except along the lines of slope $m = \pm m_0$.

Infinite Dimensions

Our current work is motivated by the oscillating beam problem introduced in Chapter I. We consider the partial differential equation

$$\ddot{u} = -u'''' - \Gamma u'' + \left[\int_0^\pi u'(s)^2\, ds\right] u'' - \mu_2 \dot{u} + \mu_1 \cos \omega_0 t \tag{4.10}$$

where Γ, μ_1, μ_2, ω_0 are constants and u is a real valued function of two variables $t \in \mathbb{R}$, $x \in [0, \pi]$, subject to the boundary conditions $u(0,t) = u(\pi,t) = u''(0,t) = u''(\pi,t) = 0$. In (4.10), a superior dot denotes differentiation with respect to t and prime differentiation with respect to x.

In (4.10) substitute $u(x,t) = \sum_{k=1}^{\infty} w_k(t) \sin kx$, multiply by $\sin nx$ and integrate from 0 to π. This yields the infinite set of ordinary differential equations

$$\ddot{w}_n = n^2(\Gamma - n^2) w_n - n^2 \left[\sum_{k=1}^{\infty} k^2 w_k^2\right] w_n - \mu_2 \dot{w}_n + 2\mu_1 \left[\frac{1-(-1)^n}{\pi n}\right] \cos \omega_0 t, \qquad n = 1, 2, \dots .$$

We see that the linear parts of these equations are uncoupled and the equations divide into two types. The equations where $1 \leq n^2 < \Gamma$ have a hyperbolic equilibrium at the origin whereas, for $n^2 \geq \Gamma$ this equilibrium is a center.

For simplicity let us assume $1 < \Gamma < 4$. Then only the equation $n = 1$ is hyperbolic while the remaining equations have a center. To emphasize this let us define $x = w_1$ and $y_n = w_{n+1}, n = 1, 2, \dots$. We now write (4.10) in the form

$$\begin{aligned} \ddot{x} = {} & a^2 x - \left[x^2 + \sum_{k=1}^{\infty} (k+1)^2 y_k^2\right] x \\ & - \mu_2 \dot{x} + \frac{4}{\pi} \mu_1 \cos \omega t, \quad n = 1, \dots, N; \end{aligned} \tag{4.11a}$$

$$\begin{aligned} \ddot{y}_n = {} & -\omega_n^2 y_n - (n+1)^2 \left[x^2 + \sum_{k=1}^{\infty} (k+1)^2 y_k^2\right] y_n \\ & - \mu_2 \dot{y}_n + 2\mu_2 \left[\frac{1-(-1)^{n+1}}{\pi(n+1)}\right] \cos \omega t \quad n = 1, 2, \dots \end{aligned} \tag{4.11b}$$

where we have defined $a^2 = \Gamma - 1$ and $\omega_n^2 = (n+1)^2\left[(n+1)^2 - \Gamma\right]$.

The reduced equation, (4.11a) with $y = 0$, is

$$\ddot{x} = a^2 x - x^3 - \mu_2 \dot{x} + \frac{4}{\pi} \mu_1 \cos \omega_0 t.$$

Once again we obtain the forced, damped Duffing equation from Example 3.1 where we showed the presence of chaos. It remains to adapt the techniques from above for a finite dimensional canter manifold to handle (4.11b).

The preceding example can be incorporated into a more general theory. Let us consider equations of the form

$$\ddot{u} = Au + \Phi(u) - \mu_2 \dot{u} + \mu_1 \cos \omega_0 t. \tag{4.12}$$

Here, μ_1, μ_2, ω_0 are constants and u is a twice differentiable function from $\mathbb{R}$ to a Hilbert space $\mathbb{X}$. A is a linear, self-adjoint operator on $\mathbb{X}$ with a finite number of positive eigenvalues and $\Phi \in \mathcal{C}^2(\mathbb{X}, \mathbb{X})$.

Let $\{\lambda_1, \lambda_2, \dots\}$ denote the eigenvalues of A numbered in decreasing order, let $\{\varphi_1, \varphi_2, \dots\}$ denote a complete set of eigenfunctions for A with φ_n corresponding to λ_n. We obtain a Bubnov-Galerkin-Kantorovich decomposition of (4.12) by substituting $u(x,t) = \sum_{k=1}^{\infty} w_k(t)\varphi_k(x)$ and then taking the inner product with φ_n for $n = 1, 2, \dots$. This yields an infinite set of ordinary differential equations. The nth equation is

$$\ddot{w}_n = \lambda_n w_n + \phi_n(w) - \mu_2 \dot{w}_n + \mu_1 A_n \cos\omega_0 t \tag{4.13}$$

where $\phi_n(w) = \langle \Phi\left(\sum_{k=1}^{\infty} w_k\varphi_k\right), \varphi_n\rangle$ and $A_n = \langle 1, \varphi_n\rangle$.

We shall assume that A has a finite number of positive eigenvalues so that there exists an integer N so that $\lambda_n > 0$ for $1 \le n \le N$ and $\lambda_n \le 0$ for $N > n$. Define $x_i(t) = w_i(t)$, $a_i^2 = \lambda_i$ for $i = 1, 2, \dots, N$ and $y_i(t) = w_{i+N}(t)$, $\omega_i^2 = -\lambda_i$ for $i = 1, 2, \dots$. Equation (4.13) now decomposes as

$$\ddot{x}_n = a_n^2 x_n + f_n(x,y) - \mu_2 \dot{x}_n + \mu_1 A_n \cos\omega_0 t, \tag{4.14a}$$

$$\ddot{y}_n = -\omega_n^2 y_n + g_n(x,y) - \mu_2 \dot{y}_n + \mu_1 B_n \cos\omega_0 t \tag{4.14b}$$

where $B_n = A_{n+N+1}$ and

$$f_n(x,y) = \phi_n\big((x_1, \dots, x_N, y_1, \dots)\big),$$
$$g_n(x,y) = \phi_{n+N+1}\big((x_1, \dots, x_N, y_1, \dots)\big).$$

We shall make this our starting point with $x \in \mathbb{R}^N$ and $y \in \mathcal{Y}$ where

$$\mathcal{Y} = \left\{ (y_{,1}, y_2, \dots) \,\middle|\, y_i \in \mathbb{R}, \sum_{n=1}^{\infty} \omega_n^2 y_n^2 < \infty \right\}.$$

We make the following assumptions about (4.14):

(H1) Each f_n, g_n is $\mathcal{C}^3$ in each argument.
(H2) $f_n(0,0) = 0$ and $g_n(x,0) = 0$.
(H3) $D_1 f_n(0,0) = 0$ and $D_2 f_n(x,0) = 0$.
(H4) $D_2 g_n(0,0) = 0$ and given $r > 0$ there exists $A > 0$ such that $\|D_2 g_n(x,0)\| \le A\omega_n$ for all n.
(H5) The system of equations $\ddot{x}_n = a_n^2 x_n + f_n(x,0)$ has a nontrivial homoclinic solution γ.

The reduced system of equations for (4.14) is

$$\ddot{x}_n = a_n^2 x_n + f_n(x,0) - \mu_2 \dot{x}_n + \mu_1 c_n \cos\omega_0 t, \quad n = 1, \dots, N.$$

This is a system of ordinary differential equation in $\mathbb{R}^N$. When written as a first order system in $\mathbb{R}^{2N}$ we get a system of equations which satisfies the hypotheses of (3.2) so that we can apply Theorem 3.2 or Theorem 3.5 to obtain chaos. It remains to extend this chaos to the full equation (4.14) using the center manifold techniques from the first part of this chapter.

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