Estimation of the amplitude of resonance in the General Standard Map

Arturo Olvera

IIMAS-UNAM, FENOMEC

Apdo. Postal 20–726, México D. F. 01000, Mexico

Abstract

The goal of this paper is to formulate some conjectures about the amplitude of resonance in the General Standard Map. The main idea of this work is to expand the periodic perturbation function in Fourier series. Given any rational rotation number, we choose a finite number of harmonics of their Fourier expansion and we compute the amplitude of resonance of the reduced perturbation function of the map, using a suitable normal form around the resonance, which is valid for asymptotically small values of the perturbation parameter. For this map, we obtain a relation between the amplitude of resonance and the perturbation parameter: the amplitude is proportional to a rational power of the parameter hence can be represented as a straight line in a log–log graph. The convex hull of these straight lines gives a lower bound for the amplitude of resonance, which is valid even when the perturbation parameter is of order one. We find that some perturbation functions give rise a phenomenon that we call collapse of resonance, this means that the amplitude of resonance goes to zero for some value of the perturbation parameter. We find an empirical procedure to estimate this value of the parameter related to the collapse of resonance.

1. Introduction.

The study of stability and the chaotic behavior of Hamiltonian systems with two degrees of freedom is a very important problem in Classical Mechanics and Dynamical Systems. The problem of finding the threshold of stability has been studied by many researchers; outstanding contributions to this problems were given by Chirikov and Escande in the beginning of the 70's.

It is well known that Hamiltonian flow can be reduced to a two dimensional map using a Poincaré section; in that form the dynamics of the Hamiltonian flow can be studied like the stability problem of the two dimensional map. In order to find the regions of stability and the threshold of chaotic behavior, we can use the Chirikov overlap method [2]. The idea of this method is to obtain the shape of the invariant manifolds of the hyperbolic periodic orbits of our map; a pictorial description of these manifolds looks like a chain of pendulum separatrices (see figure 1). It is common to call resonances the structures which are defined by the hyperbolic periodic points and their invariant manifolds (or separatrices). The resonances are denoted by the rotation number of the corresponding periodic orbit. Chirikov studied the interaction of two resonances; in the case that the separatrices of these two resonances overlap, then we can expect to find chaotic behavior around the resonances. A similar idea has been developed by Escande [4] and Simó and Olvera [8].

One important aspect of the overlap method is the necessity to estimate the amplitude of the resonances. This amplitude corresponds to the maximum distance of the separatrices (in figure 1, the maximum distance is given in the vertical direction). We denote this amplitude by $\Delta_{p/q}$, where the subindex p/q is the rotation number of the periodic orbit, and this is a rational number.

In many cases, the two dimensional map can be described as the sum of an integrable twist map and a small perturbation. The amplitude of the perturbation is driven by one real parameter. The goal of this paper is to obtain a simple relation between the perturbation parameter and the amplitude of resonance for any rational rotation number.

In this work we choose a particular twist map that is known as the Standard map. This kind of two dimensional map is very important in the study of Hamiltonian systems with two degrees of freedom; Chirikov [2] and Lichtenberg [5] describe many examples of how we can reduce the Hamiltonian flow in order to obtain a Standard map. In this case, when we are interested in studying the dynamics around any fixed point (or periodic orbit) of any Hamiltonian system with two degrees of freedom, it is easy to find the corresponding twist map which represents the dynamics around the fixed point. This procedure is described in appendix A of this paper.

Now, we are going to define the General Standard Map and study the resonances of this map and their relation with the perturbation parameter. This map is defined in the following way:

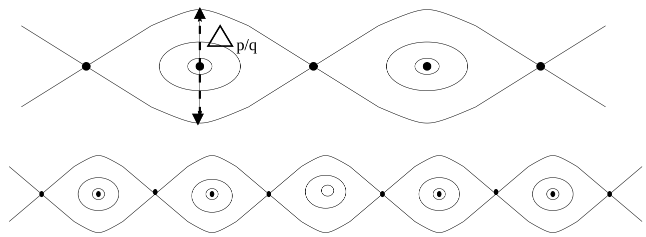

Figure 1: Phase space of two resonances.

$$\begin{aligned} \rho_{i+1} = & \quad \rho_i + \epsilon V(\phi_i, \epsilon) \quad , \\ \phi_{i+1} = & \quad \phi_i + \rho_{i+1} \quad . \end{aligned} \tag{1}$$

The radial variable $\rho_i \in \mathbb{R}$ and the angular variable $\phi \in \mathsf{S}^1$. The perturbation function $V(\phi, \epsilon)$ is analytic and periodic respect to the angular variable, that is, $V(\phi + 1, \epsilon) = V(\phi, \epsilon)$. The perturbation parameter ϵ is a non negative real number. For $\epsilon = 0$, the map (1) is reduced to an integrable twist map and the phase space ($\mathbb{R} \times \mathsf{S}^1$) is completely foliated by invariant circles. For any value of the radial variable ρ, there exists an invariant circle with rotation number ρ.

For a small perturbation, $0 < \epsilon << 1$, we can use the Birkhoff twist theorem to show that the set of invariant circles with rational rotation number disappears (in the generic sense), and it is transformed into an even number of periodic orbits with the same rational rotation number. The linear stability of these orbits is elliptic and hyperbolic. For asymptotically small values of the parameter, it is possible to find the amplitude of any resonance (in the generic case) of the corresponding hyperbolic orbit. It is well known that this amplitude is of the form $\Delta_{p/q} \sim \epsilon^{1/2}$. The way to obtain this relation is described in appendix B.

The main contribution of this paper is to find the way in which the amplitude of resonance depends on the perturbation parameter when its value is not asymptotically small; we can describe the amplitude of resonance when $\epsilon \in \mathcal{O}(1)$. We are going to show that the behavior of the amplitude of resonance is not homogeneous in all the domain of ϵ, we find that there are many intervals of the value of the parameter ϵ where the behavior of the amplitude of resonance is different respect to other intervals. Let the interval $(\epsilon_i, \epsilon_{i+1})$, then the form of the amplitude of resonance is $\Delta_{p/q} = A_i \epsilon^{n_i}$, so we have different values of the rational exponent n_i for each interval of the domain of ϵ. In this form we can determine the values of ϵ_i which correspond to the border values where amplitude of resonance changes its rate of growth.

Our procedure to find the amplitude of resonance, $\Delta_{p/q}$, for any value of the perturbation parameter can be divided in four steps.

1. First, we get the Fourier expansion of the perturbation function $V(\phi, \epsilon)$ and we define a new perturbation function $V_N(\phi, \epsilon)$ which corresponds to

selecting some of the the first N harmonics of $V(\phi, \epsilon)$ (where $N \geq q$).

2. Next, we use $V_N(\phi, \epsilon)$ as the perturbation function of the Standard Map (1). Now, we compute an adequate normal form. This normal form is obtained by the use of the Poincaré Lindstedt method. We use this normal form to determine the dynamics close to the periodic orbit which looks like the periodic orbits of the pendulum equation. Therefore, it is easy to find the amplitude of the separatrices of this pendulum equation. The normal form that we obtain is only valid for asymptotically small values of ϵ.

3. We repeat the first two steps several times, changing the set of harmonics that we select to form the function $V_N(\phi, \epsilon)$.

4. We find that the behavior of $\Delta_{p/q}$, for a large value of ϵ, is the direct sum of the amplitudes of resonance that we got in the previous steps.

We conclude that only a few harmonics of the Fourier expansion of $V(\phi, \epsilon)$ are responsible for the behavior of $\Delta_{p/q}$ in each interval of the domain of ϵ. We must remark that we can predict the size of $\Delta_{p/q}$ for a large value of ϵ using only asymptotic information obtained from the normal forms that we compute in the Standard Map using different perturbation functions $V_N(\phi, \epsilon)$.

The structure of this paper is the following: Section 2 describes the method to obtain the normal form for perturbation functions which are defined as a trigonometric polynomial of ϕ. This normal form only depends on the rotation number and the set of harmonics that form the perturbation function. The next section is devoted to showing how we can get this kind of normal form using some information of the linear stability of the hyperbolic periodic orbit. In order to compare our estimate of $\Delta_{p/q}$, we develop a numerical procedure to compute this amplitude of resonance and this procedure is shown in section 4. We select some examples in section 5 to compare the numerical computation respect to our method for estimating the amplitude of resonance. In the last section we describe one interesting phenomenon related to the amplitude of resonance which we call collapse of resonance; this means that for some values of ϵ the amplitude of resonance goes to zero.

2. Resonant normal forms.

The main task in this section is to work out a procedure to obtain a map which displays in a simple form the dynamics around a specific periodic orbit. The idea is to perform a set of coordinate transformations in order to obtain a new map which must be closer to an integrable map (in the neighborhood of the given rotation number). We are going to show that we need only perform a finite number of these transformations if we set the rotation number as a rational number p/q with $(p, q) = 1$. The final map can be related to a simple

Hamiltonian flow. The dynamics of the Hamiltonian system is close to the final map up to some order of the perturbation parameter ϵ. Our procedure is similar to the Lindstedt series method, which is a standard procedure to solve nonlinear differential equations using asymptotic methods in Celestial Mechanics. In order to perform symplectic transformations we must rewrite our map, we take the two difference equations of (5) and we rewrite it like one second order difference equation; we call this new equation the Lagrangian representation of the Standard Map.

Consider the map (1) where the perturbation function $V(\phi, \epsilon)$ is an analytic periodic function with null average, such that $V(\phi+1, \epsilon) = V(\phi, \epsilon)$. A general expression of this function is given by the Fourier expansion of the angular variable, the coefficient of each Fourier term is a power series of the parameter ϵ:

$$V(\phi, \epsilon) = \sum_{s=1}^{\infty} \sum_{j=-\infty}^{\infty} C_{s,j} \epsilon^s e^{2\pi i j \phi} \quad , \tag{2}$$

We can obtain a good approximation of the perturbation function taking only the first N harmonic terms of equation (2), where $N \leq q$. Each harmonic is multiplied by the leading term of the corresponding power series of the perturbation parameter. Our approximation is then given in this form:

$$V(\phi, \epsilon) = \sum_{j=1}^{N} \epsilon^{a_j} \left(c_j e^{2\pi i w_j \phi} + \hat{c}_j e^{-2\pi i w_j \phi} \right) + \mathcal{O}(\epsilon^{S_N}) \quad , \tag{3}$$

where a_i are positive integer numbers, the frequencies w_i are positive integer numbers such that $w_i < w_{i+1}$ and c_i, $\hat{c}_i$ are complex numbers which do not depend on ϵ. The remaining term $\mathcal{O}(\epsilon^{S_N})$, where $S_N > \max_{j=1,..,N}\{a_j\}$, does not contain harmonics of order less than w_N. We assume that the magnitude of the coefficients of the harmonic terms, which belong to $\mathcal{O}(\epsilon^{S_N})$, are smaller than $\min_{j=1,..,N}\{c_j, \hat{c}_j\}$. If $V(\phi, \epsilon)$ is analytic then we can see that these coefficients c_j and $\hat{c}_j$ are proportional to γ^j where γ is a constant such that $|\gamma| < 1$ and $a_j = 0$.

In order to find the resonant normal form we must take the lift of our map (1). For this map, the phase space is now IR^2. We can write the lift in the following form:

$$\begin{aligned} y_{i+1} = &\quad y_i + \epsilon V(x_i, \epsilon) \quad , \\ x_{i+1} = &\quad x_i + y_{i+1} \quad , \end{aligned} \tag{4}$$

where $(x_i, y_i) \in \mathrm{IR}^2$ and $i \in \mathbb{Z}$. This map is a set of two difference equations of first order; we can rewrite it as a single difference equation of second order, and this is the Lagrangian form of (1):

$$x_{i+1} - 2x_i + x_{i-1} = \epsilon V(x_i, \epsilon) \quad . \tag{5}$$

We want to find a coordinate transformation which conjugates, as far as possible, the dynamics of equation (5) to the dynamics of an integrable twist map in the neighborhood of a given rotation number p/q. The dynamics are given as an uniform rotation:

$$\theta_{i+1} = \theta_i + \frac{p}{q} \quad , \tag{6}$$

where $\theta_i \in \mathbb{R}$.

Let the function $g(\theta, \epsilon)$ define the conjugation between the coordinate x and the coordinate θ:

$$x_i = \theta_i + g(\theta_i, \epsilon) \quad . \tag{7}$$

The function $g(\theta, \epsilon)$ is periodic and we can expand it as a power series in ϵ where the coefficients are periodic functions of θ and they are given by the Fourier series:

$$g(\theta, \epsilon) = \sum_{j=0}^{\infty} \epsilon^j \sum_{k \in \mathbb{Z}} g_{j,k} \mathrm{e}^{2\pi i k \theta} \quad . \tag{8}$$

We can replace (8) and (7) in (5), taking into account that the dynamic of θ is an uniform rotation (6), then we obtain the following relation:

$$\begin{aligned} &\theta_{i+1} - 2\theta_i + \theta_{i-1} + \sum_{j=0}^{\infty} \epsilon^j \sum_{k \in \mathbb{Z}} g_{j,k} \mathrm{e}^{2\pi i k \theta} 2 \left(\cos(2\pi \frac{jp}{q}) + 1 \right) = \\ &= \sum_{i=1}^{N} \epsilon^{a_i} \left(c_i^+ \mathrm{e}^{2\pi i w_i (\theta_i + g(\theta, \epsilon))} + c_i^- \mathrm{e}^{-2\pi i w_i (\theta_i + g(\theta, \epsilon))} \right) \end{aligned} \tag{9}$$

The right hand side of equation (9) is an exponential function whose arguments include the function $g(\theta, \epsilon)$, so we can expand the exponential function as a power series of $g(\theta, \epsilon)$:

$$\mathrm{e}^{\pm 2\pi i w_i (\theta + g(\theta, \epsilon))} = \mathrm{e}^{\pm 2\pi i w_i \theta} \left[1 + \sum_{s=1}^{\infty} \frac{1}{s!} \left(\pm \sum_{j=0}^{\infty} \epsilon^j \sum_{k \in \mathbb{Z}} g_{j,k} \mathrm{e}^{2\pi i k \theta} \right)^s \right] \quad . \tag{10}$$

Substituting (10) into (9) and considering that $\theta_{i+1} - 2\theta_i + \theta_{i-1}$ must be 0 (because the dynamic of θ is an uniform rotation), then:

$$\begin{aligned} &\sum_{j=0}^{\infty} \epsilon^j \sum_{k \in \mathbb{Z}} g_{j,k} \mathrm{e}^{2\pi i k \theta} 2 \left(\cos(2\pi \frac{jp}{q}) - 1 \right) = \\ &= \sum_{i=1}^{N} \epsilon^{a_i} \left(c_i^{\pm} \mathrm{e}^{\pm 2\pi i w_i \theta} \left[1 + \sum_{s=1}^{\infty} \frac{1}{s!} \left(\pm \sum_{j=0}^{\infty} \epsilon^j \sum_{k \in \mathbb{Z}} g_{j,k} \mathrm{e}^{2\pi i k \theta} \right)^s \right] \right) \end{aligned} \quad . \tag{11}$$

In order to determine the values of the coefficients $g_{j,k}$ we must solve equation (11). For any positive integer number m we can collect all the terms of order $\mathcal{O}(\epsilon^m)$; these terms depend on the coefficients $g_{j,k}$, where $j = 0, .., m-1$ and k is a finite subset of $\mathbb{Z}$. The coefficient $g_{m,k}$ appears in a linear form in the left side of equation (11). Then we have an infinite number of homological equations, each equation is related to the same order of the perturbation parameter, ϵ^m, and a specific harmonic term $\mathrm{e}^{2\pi ik\theta}$. This homological equation can be expressed in the following way:

$$2g_{m,k}\left(\cos(2\pi\frac{kp}{q})-1\right)=G_{m,k}(g_{0,k'},g_{1,k''},...,g_{m-1,k^{(m)}}) \tag{12}$$

where $k,k',...,k^{(m)}\in\mathbb{Z}$. We can solve (12) because its right hand side depends only on the coefficients $g_{i,k}$ that we have already computed. We must remark that for fixed order of ϵ, the right hand side of (11) has a finite number of harmonic terms, such that, for any order of ϵ we must solve a finite number of homological equations.

There are two conditions that we must satisfy if we want to solve these homological equations (12):

1. For any order of ϵ, the right side of equation (11) must have null average.
2. The term $\cos(\frac{kp}{q})-1$ must be different from zero.

The first condition means that for any value of m in equation (12) the term $G_{m,0}(g_{0,k},..,g_{m-1,k^{(m)}})$, which corresponds to a harmonic term of zero frequency, must be zero in order to have null average. As a consequence, the coefficients of type $g_{m,0}$ are indeterminate and we can fix any value for them. We can use these free parameters to ensure that $G_{m,0}(g_{0,k},..,g_{m-1,k^{(m)}})=0$.

The second condition is the non–resonance condition. If the value of k is $\pm q$ then $\cos(2\pi\frac{kp}{q})-1=0$. In this case there are no solutions of the homological equation (12). Therefore we cannot continue the procedure to solve the homological equation for the next order of ϵ. This condition defines the resonant normal form. There are some cases where $G_{m,\pm q}=0$, in this situation the homological equation is trivial and we can continue our procedure until we can reach the next non–resonance condition when $k=2q$.

We can solve only a finite number of homological equations (12) because there exists an integer number $\bar{m}$ such that the non–resonant condition is not satisfied. For this value of $\bar{m}\in\mathbb{N}$ there are terms of order $\mathcal{O}(\epsilon^{\bar{m}})$ which contain harmonic terms with frequency $\pm q$ (and $G_{\bar{m},\pm q}(g_{j,k})\neq 0$ in general). As a consequence, it is impossible to conjugate the dynamics of our map (5) to the dynamics of an uniform rotation in the neighborhood of the rotation number p/q. Therefore $\theta_{i+1}-2\theta_i+\theta_{i-1}=\mathcal{O}(\epsilon^{\bar{m}})\neq 0$ for $i\in\mathbb{Z}$. Hence, we say that this equation is a resonant normal form and the dynamics of the coordinate θ are then described in this form:

$$\theta_{i+1}-2\theta_i+\theta_{i-1}=\epsilon^{\bar{m}}G_{\bar{m},\pm q}(g_{j,k})\mathrm{e}^{\pm 2\pi iq\theta_i}+\mathcal{O}(\epsilon^{\bar{m}+1}) \quad . \tag{13}$$

We can shift the origin of the coordinate θ and we can rewrite equation (13) in the form:

$$\theta_{i+1}-2\theta_i+\theta_{i-1}=\epsilon^{\bar{m}}A_{p/q}\sin(2\pi q\theta_i)+\mathcal{O}(\epsilon^{\bar{m}+1}) \quad . \tag{14}$$

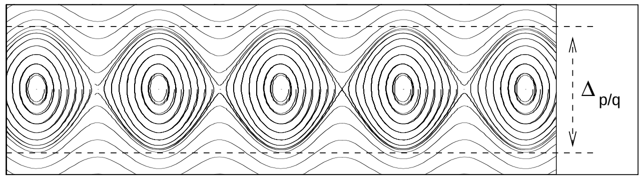

Figure 2: Phase space of the q-pendulum equation.

We divide equation (14) by $\epsilon^{\bar{m}}$, then we can interpret the left side as the approximation of a second derivative of the variable θ respect to time, where $\Delta t = \sqrt{\epsilon^{\bar{m}}}$. The related differential equation is the pendulum equation:

$$\frac{d^2\theta}{dt^2} = A_{p/q} \sin(2\pi q \theta) \quad . \tag{15}$$

The phase space of this equation is shown in figure (2). The phase space of our resonant normal form (14) is similar to figure (2) to order $\epsilon^{\bar{m}+1}$. We can see that there are $2q$ periodic points which correspond to the image of two periodic orbits, the first one has elliptic linear stability and the second one is hyperbolic.

The maximum amplitude of the separatrices of equation (15) is given by $\Delta = 4\sqrt{A_{p/q}/(2\pi q)} = 4\lambda/(2\pi q)$ where λ is the eigenvalue of the hyperbolic point. The corresponding value of the amplitude of resonance related to the resonant normal form (14) is then:

$$\Delta_{p/q} = \epsilon^{\bar{m}/2} \sqrt{\frac{A_{p/q}}{2\pi q}} + \mathcal{O}(\epsilon^{\frac{\bar{m}+1}{2}}) \quad , \tag{16}$$

and the eigenvalues of the hyperbolic points are given by:

$$\lambda_{p/q} = 1 \pm \epsilon^{\bar{m}/2} \sqrt{2\pi q A_{p/q}} + \mathcal{O}(\epsilon^{\frac{\bar{m}+1}{2}}) \quad . \tag{17}$$

There are other alternative ways to work out the resonant normal forms related to periodic orbits. It is possible to compute the image of the q-th iteration of map (5), we can carry out the composition of the trigonometric function using the properties of Bessel functions. Given the explicit computation of F^q we can use a fixed-point-like procedure to obtain the resonant normal form. This procedure is shown in appendix 3.

3. Rules to determine the resonant normal form.

The amplitude of resonance of any hyperbolic periodic orbit with rotation number p/q can be obtained if we compute its resonant normal form. In general, it is a hard task to solve a large sequence of homological equations in order to obtain the value of the exponent $\bar{m}$ and the coefficient $A_{p/q}$. Nevertheless, there is an alternative procedure for finding the value of $\bar{m}$ in a easy way. We must follow these steps:

1. From equation (3) we can collect the set of exponents $\{a_i\}$ and the corresponding frequencies $\{w_i\}$, where $i = 1, .., N$.
2. Let $\{n_i\}$ be any set of N integer numbers which has the follow property:
$$q = \sum_{i=1}^{N} n_i w_i \quad .$$
3. Let m be a natural number which is obtained in this form:
$$m = \sum_{i=1}^{N} |n_i| a_i \quad .$$
4. In this way, we can find the set of integers $\{n_i\}$ which minimize the value of m. This minimum m correspond to the value of the exponent $\bar{m}$. In this case we must suppose that $A_{p/q} \neq 0$.

In order to show how this algorithm can give the value of $\bar{m}$, we follow the construction of the resonant normal form step by step: From equation (3), we have the set of exponents $\{a_i\}$, $i = 1, .., N$. This set can be ordered and written in the form, $\{\hat{a}_i\}$, $i = 1, .., N$, where $\hat{a}_i < \hat{a}_{i+1}$. Let the set of corresponding harmonic frequencies be $\{\hat{w}_i\}$. The left side of the homological equation (11) can be expressed as a power series in ϵ, we can see that each term of this series will be a product of terms $(\epsilon^j g_{j,k} \mathrm{e}^{\pm 2\pi i k\theta})^n$. The lowest order term of this series is $\epsilon^{\hat{a}_1} g_{\hat{a}_1,\hat{w}_1} \mathrm{e}^{\pm 2\pi i \hat{w}_1 \theta}$. The next term could be $\epsilon^{\hat{a}_2} g_{\hat{a}_2,\hat{w}_2} \mathrm{e}^{\pm 2\pi i \hat{w}_2 \theta}$ or any product of the first term if $\hat{a}_2 \geq n\hat{a}_1$, where n is a positive integer. In this case the corresponding harmonic frequency is given by $\hat{w}_2$ or $\pm n\hat{w}_1$. The following term could be $\epsilon^{\hat{a}_3} g_{\hat{a}_3,\hat{w}_3} \mathrm{e}^{\pm 2\pi i \hat{w}_3 \theta}$ or any product of previous term if $\hat{a}_3 \geq n\hat{a}_1 + m\hat{a}_2$, where n and m are positive integers. The related harmonic frequency is then given by $\hat{w}_3$ or $\pm n\hat{w}_1 \pm m\hat{w}_2$. We can continue this procedure in a similar way to find the higher order terms of this power series.

We can see from the previous procedure that any harmonic frequency w is going to appear the first time when we can find a linear combination of the harmonic terms $\{w_1, w_2, .., w_N\}$ such that $w = \sum_{j=1}^{N} n_j w_j$. The set of integers

n_j is chosen to minimize the following sum, $m_w = \sum_{j=1}^{N} |n_j| a_j$. Given the value of m_w we know that the harmonic terms of frequency w are of order $\mathcal{O}(\epsilon^{m_w})$.

To compute the value of $A_{p/q}$ we must find the resonant normal form in an explicit form. It is necessary to use an algebraic manipulator to find this normal form. An alternative procedure to estimate the value of $A_{p/q}$ for any resonant normal form is the following: we can use a numerical procedure to compute the eigenvalues of the hyperbolic periodic points with rotation number p/q, we must compute these eigenvalues when the value of the parameter goes to zero. Given the values of $\lambda_{p/q}$ and $\bar{m}$, then we can find the value of $A_{p/q}$ using equations (16) and (17). The numerical algorithm to find the hyperbolic periodic points can be reduced to a one dimensional method if our periodic orbits are monotone orbits.

4. Numerical computation.

In the last section we developed asymptotic methods to obtain the amplitude of any resonance. Now, we want to compare these results with the numerical computation of the amplitude of resonance. From a formal point of view, asymptotic methods can be applied for a very small value of the perturbation parameter, therefore the numerical methods must be implemented with high precision arithmetic because the linear behavior close to the periodic orbits has extremely slow dynamics since the eigenvalues of the periodic points are proportional to a power of the perturbation parameter (see equation 17).

The main idea to compute the amplitude of resonance is to find the invariant manifolds of the hyperbolic periodic orbits related to our resonance. A naive method to compute the invariant manifolds is finding the eigenvectors of the periodic orbits; then we can iterate points which are located in the direction of the eigenvectors but these points must be chosen very close to the periodic point. In this way we can expand the unstable invariant manifold iterating these points a sufficient number of times (for the stable manifold we use the inverse map). Nevertheless, this method is very inefficient for small values of the parameter because we must perform a great number of iterations in order to escape from the neighborhood of the periodic orbit.

An alternative method to solve this problem is to find a better approximation of the local invariant manifolds, representing them as the graph of a polynomial function. In that way we can choose points which belong to the graph and they can be located far from the periodic point. The computation of the local invariant manifolds gives a numerical method to obtain the resonance of any periodic point. We can use this method to find the maximum distance between the manifolds (this distance is measured in the orthogonal direction of the straight line that joins the periodic point with the next image of this point), in that way we obtain a numerical computation of the amplitude of resonance.

Now, we are going to describe how we can compute the invariant manifolds and the amplitude of resonance:

1. Finding periodic orbits: The first problem is to find a periodic orbit with specific rotation number p/n. This is a two–dimensional problem, then we must use a two dimensional root finder to locate the periodic orbit. Nevertheless we can consider some important properties of our map which are related to the symmetries of these kinds of maps. That is, any two–dimensional map which can be written as the product of two involutions (a map A is an involution if $A \circ A = \mathrm{Id}$). In this case, we are able to find periodic orbits by looking for some specific curves which correspond to the invariant curves of the involutions; these curves are known as symmetry lines of our map. This is a useful property of this kind of map because we can reduce the problem of finding periodic orbits to a one dimensional problem. A complete description of this procedure can be found in reference [3].

2. Finding invariant manifolds: Once the periodic orbit has been found, we can compute the eigenvalues and the eigenvectors of this point. In this way, we are able to determine the local invariant manifolds. A first approximation of these invariant sets is given by the segment of a straight line which starts from the periodic point and is parallel to the corresponding eigenvector. This linear approximation is not enough for our purposes, we must find a high order approximation. Let us denote our map in the following way:

$$\begin{aligned} x_{n+1} &= F(x_n, y_n) \quad , \\ y_{n+1} &= G(x_n, y_n) \quad . \end{aligned} \tag{18}$$

Let (x_p, y_p) be our periodic point with rotation number p/n, then it is a fixed point of the q–th composition of our map:

$$\begin{aligned} x_p &= F^n(x_p, y_p) \quad , \\ y_p &= G^n(x_p, y_p) \quad . \end{aligned} \tag{19}$$

Now we must find the Taylor expansion of the functions F^n and G^n around the periodic point. Let $\widetilde{F}(\hat{x}, \hat{y})$ and $\widetilde{G}(\hat{x}, \hat{y})$ be the Taylor series of the map (18), where the new variables are $\hat{x} = x - x_p$ and $\hat{y} = y - y_p$. We can compute the second iteration of (18) using our polynomial approximation:

$$\begin{aligned} \widetilde{F}^2(\hat{x}, \hat{y}) &= \widetilde{F}\left(\widetilde{F}(\hat{x}, \hat{y}), \widetilde{G}(\hat{x}, \hat{y})\right) \quad , \\ \widetilde{G}^2(\hat{x}, \hat{y}) &= \widetilde{G}\left(\widetilde{F}(\hat{x}, \hat{y}), \widetilde{G}(\hat{x}, \hat{y})\right) \quad . \end{aligned} \tag{20}$$

Then, we can compute the Taylor expansion of (20). We know that $\widetilde{F}$ and $\widetilde{G}$ are polynomials of degree M, then the composition $\widetilde{F}\left(\widetilde{F}(\hat{x}, \hat{y}), \widetilde{G}(\hat{x}, \hat{y})\right)$

should be computed as a polynomial of degree M. It can be done by the use of symbolic computational routines which can calculate the sum and product of two polynomials; these routines can truncate the polynomial expansion up to degree M.

In this way we obtain an expression for the map around the periodic point which can be written as a polynomial of degree M. We can repeat this procedure n–times in order to obtain a polynomial approximation of the map (19). In our new set of coordinates, the origin is a fixed point of the map:

$$\begin{aligned} \hat{x}_{n+1} &= \widetilde{F}^n(\hat{x}_n, \hat{y}_n) \quad , \\ \hat{y}_{n+1} &= \widetilde{G}^n(\hat{x}_n, \hat{y}_n) \quad . \end{aligned} \tag{21}$$

The next step is to describe the invariant manifolds as the graph of a function of one variable $\phi(\hat{x}) = \hat{y}$. This function can be approximated by a polynomial of degree M. Let $\phi(\hat{x})$ be the graph which describes the local invariant manifold; we know that the image of any point of the invariant manifold must belong to the invariant manifold. Taking into account this property of the invariant manifold, we obtain the following relation:

$$\phi\left(\widetilde{F}^n(\hat{x}, \phi(\hat{x}))\right) = \widetilde{G}^n(\hat{x}, \phi(\hat{x})) \quad . \tag{22}$$

Because $\widetilde{F}^n$ and $\widetilde{G}^n$ are polynomials of degree M then we can manipulate equation (22) in order to determine the value of the unknown coefficient which defines the polynomial function $\phi(\hat{x})$ up to degree M.

We define now the fundamental interval; this set belongs to the unstable invariant manifold and it can be described as:

$$\overline{FI} = \left\{(\hat{x}, \phi(\hat{x}))| \quad \text{where} \quad \hat{x} \in [\hat{x}_0, \widetilde{F}^n(\hat{x}_0, \phi(\hat{x}_0))]\right\} \tag{23}$$

where $|\hat{x}_0| << 1$. For the stable invariant manifold we substitute the map by $\widetilde{F}^{-n}$ in (23). The successive images of the fundamental interval form the local invariant manifold. We want to extend the invariant manifold until this curve arrives close to the next image of our periodic orbit. In order to select a suitable value of $\hat{x}_0$, we must take in to account that the residue of $\phi(\hat{x})$ is of order $\mathcal{O}(\hat{x}_0^{M+1})$, this approximation allows us to choose values of $\hat{x}_0$ greater than the linear approximation of the invariant manifold in some order of magnitude.

3. Amplitude of resonance: To find the amplitude of resonance, we set the periodic point $\mathbf{x}_0$ and we also find the nearest image of this point in phase space, this point is denoted by $\mathbf{x}_1$. Now, we have computed the approximation of the stable invariant manifold of our periodic point, $\mathcal{W}^s(\mathbf{x}_0)$,

and the unstable one, $\mathcal{W}^u(\mathbf{x}_0)$. Let $\overline{SP}$ be the line segment that joins the points $\mathbf{x}_0$ and $\mathbf{x}_1$, this segment is parametrized by the function $sp(t)$, where $t \in [0,1]$. We define $\overline{SW}(t)$ as the line segment which is orthogonal to $\overline{SP}$ at the point $sp(t)$, and has initial point located in $\mathcal{W}^s(\mathbf{x}_0)$ and its final point in $\mathcal{W}^u(\mathbf{x}_0)$. The amplitude of resonance is defined as the maximum length of $\overline{SW}(t)$ for $t \in [0,1]$.

A more complete description of the construction of invariant manifolds using numerical and asymptotic methods can be found in reference [10].

An important question for any resonance is to find how the amplitude of resonance depends to the perturbation parameter ϵ. In order to obtain this relation, we can compute the amplitude of resonance for a sequence of values of the parameter $\{\epsilon_i\}$, such that $\epsilon_{i+1} < \epsilon_i$, for $i = 0, \ldots, L$. Given any value of ϵ_i, we can compute the corresponding amplitude of resonance Δ_i. After that, we can compute a linear regression of the set of values $(\log(\epsilon_i), \log(\Delta_i))$, for $i = 0, \ldots, L$. The linear regression gives us a functional relation between the parameter and the amplitude of resonance:

$$\Delta = \beta \epsilon^{\alpha} \quad . \tag{24}$$

Our numerical procedure allows us to compute the invariant manifolds with a very high precision, which depends on the order of the polynomial function $\phi(\hat{x})$. In order to compute this function with an error less than 10^{-22}, the polynomial function was obtained up to degree 30 and we had to perform our numerical procedures using quadruple precision arithmetic in our computer.

At this point, we can compare the amplitude of any resonance computing the resonant normal form equation (16) and the numerical value of this amplitude. In this way we can obtain some estimate about the accuracy of the asymptotic procedure of our resonant normal form described in section 2. For this purpose, we compute the amplitude of resonance for three rotation numbers; in these examples the perturbation function has only one harmonic, $\epsilon f(x,\epsilon) = \frac{\epsilon}{2\pi}\sin(2\pi x)$ in equation (3). Table 1 shows the value of the amplitude of resonance. We compute the resonant normal form (16), also we find the numerical approximation of the coefficient Δ and β in equation (24). We can see that the accuracy of the exponent β and the coefficient Δ is good.

The resonant normal form (16) is not easy to obtain for periodic orbits with large period. This is because the number of terms that we must work out from this asymptotic method increases like a combinatorial problem. Nevertheless, we showed at the end of section 3 a procedure to estimate the amplitude of resonance without computing the complete resonant normal form. This is an asymptotic method and it is valid for asymptotically small values of the perturbation parameter. For any perturbation function (3) and rotation number p/q, we can compute the corresponding amplitude of resonance if we are able to find the eigenvalues of the hyperbolic periodic points using equations (16) and (17). In order to obtain the value of $\bar{m}$, we can follow the rules described in section

p/n	$\triangle_{p/n}$ normal form	$\triangle_{p/n}$ numerical
1/3	$\frac{\sqrt{2}}{12\pi}\,\epsilon^{3/2} \approx 0.037513\,\epsilon^{3/2}$	$0.037538\,\epsilon^{1.5009}$
1/4	$\frac{\sqrt{15}}{48\pi}\,\epsilon^{2} \approx 0.025685\,\epsilon^{2}$	$0.025684\,\epsilon^{1.9999}$
1/6	$\frac{1}{2\pi}\sqrt{\frac{33}{1280}}\,\epsilon^{3} \approx 0.02555\,\epsilon^{3}$	$0.02549\,\epsilon^{2.996}$

Table 1:

4. It is easy to find the numerical value of $\lambda_{p/q}$ for small values of ϵ. The first step of our numerical procedure describes how we can find hyperbolic periodic points and their eigenvalues. Hence, we can estimate the coefficient $A_{p/q}$ of the resonant normal form:

$$A_{p/q} = \epsilon^{-\bar{m}}\,\frac{(\lambda_{p/q}-1)^2}{2\pi q} \quad . \tag{25}$$

We must remark that it is necessary to be careful about how small we must choose the value of the perturbation parameter $|\epsilon|$ for periodic orbits with large period. This is because the numerical procedure which computes the eigenvalue of any orbit can reach an arithmetic underflow in our machine and then we can loose the precision of the numerical computation.

In the next sections we show many examples where we had to compute the amplitude of resonance; in all these cases we used the method to compute the eigenvalues in order to estimate the coefficients Δ in equation (24).

5. Estimate of $\Delta_{p/q}$ for values of ϵ of order 1.

In this section we show how we can use the asymptotic behavior of the amplitude of resonances in order to find a good estimate of $\Delta_{p/q}$ when ϵ is of order 1. The idea is that only few harmonics of the perturbation function (3) provide the main contribution to the size of the amplitude of resonance. We found that the asymptotic behavior of different sets of harmonics determine the amplitude for specific intervals of values of ϵ which can be far from 0.

We present two simple examples where we show the form that we can determine, for any range of values of ϵ, which is the dominant harmonic of (3). From these examples we express a more general conjecture about the behavior of the amplitude of resonance.

First example: Let the perturbation function have the following form:

$$f(x) = \frac{\epsilon}{2\pi}\left(\sin(2\pi x) + 10^{-\alpha}\sin(2\pi(9x))\right) \quad , \tag{26}$$

where α will be taken equal to 9, 15 and 21. Suppose that we want to find the rate of growth $\Delta_{2/9}$ for a periodic orbit with rotation number $\frac{2}{9}$. Using the rules given in section 3, we can see that the value of $\bar{m}$ must be 1. Then the asymptotic behavior can be determined with the help of equation (16). This relation is similar for the three values of the exponent α (but $A_{2/9}$ depends on the value of α). Nevertheless, the numerical computation of $\Delta_{2/9}$ shows that the asymptotic behavior is not accurate for values of ϵ greater than 10^{-3}. The results of this computation are given in figure 3. This picture shows the value of $\log_{10}(\epsilon)$ in the horizontal axis and $\log_{10}(\Delta_{2/9})$ in the vertical axis. There are three curves, F1, F2 and F3, which correspond to the value of α equal to 9, 15 and 21 respectively. For small values of ϵ, such that $|\epsilon| < 10^{-3}$, the three curves look like straight lines with slope equal to 1/2. Hence the asymptotic estimate of $\Delta_{2/9}$ agrees with the numerical computation (lines B, C and D). But we can observe that, for higher values of ϵ, each curve changes its slope to the value 9/2. This means that the rate of growth of the amplitude is now $\Delta_{9/2} \sim \epsilon^{9/2}$. For values of ϵ greater than 0.1, the three curves behave like straight lines with slope 9/2.

Now we can compare the previous results with a similar example but in this case we are going to take into account only the first harmonic of the perturbation function:

$$f(x) = \frac{\epsilon}{2\pi}\left(\sin(2\pi x)\right) \quad .$$

The asymptotic behavior of this case corresponds to a straight line with slope 9/2 (line A in figure 3). Next, we can do a similar asymptotic computation but now with a perturbation function which has only the second harmonic of (26):

$$f(x) = \frac{\epsilon}{2\pi}10^{-\alpha}\sin(2\pi(9x)) \quad ; \tag{27}$$

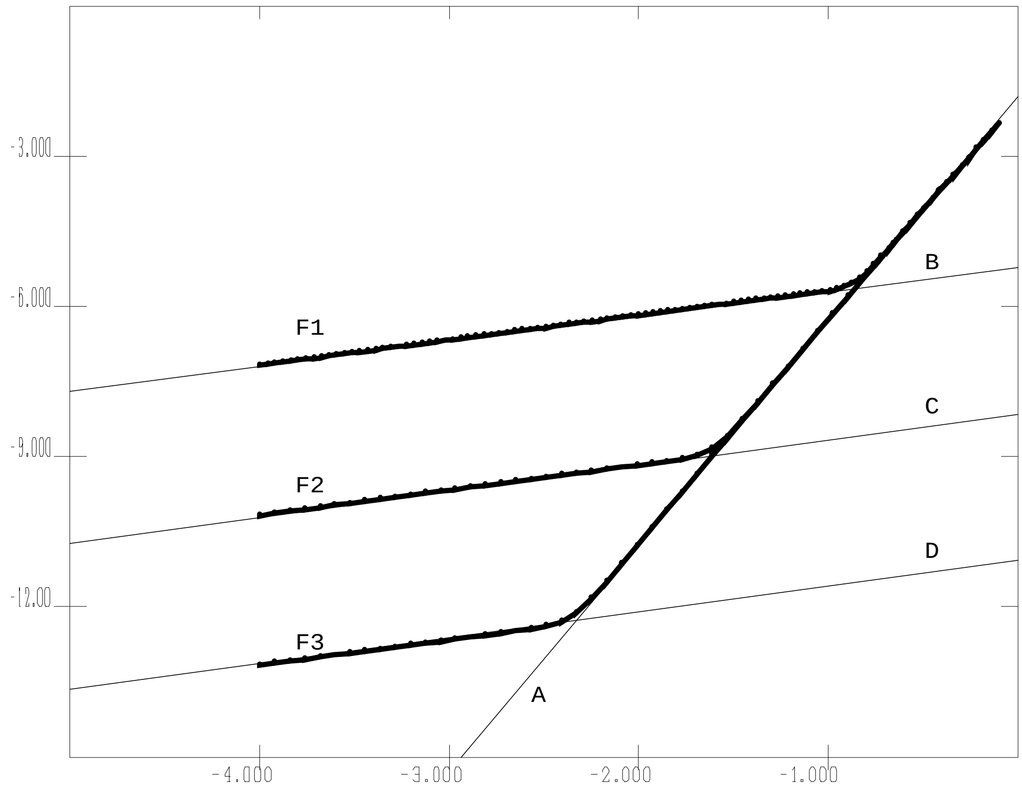

Figure 3. The horizontal axis correspond to $\log_{10}(\epsilon)$, vertical axis correspond to $\log_{10}(\Delta_{2/9})$. Bold lines F1, F2 and F3 correspond to $\Delta_{2/9}$ with α equal to 9, 15 and 21 respectively. Straight lines B, C and D are the asymptotic behavior of (26) with α equal to 9, 15 and 21 reps. Straight line A is obtained using the first harmonic in the perturbation function.

we obtain three straight lines with slope 1/2 which correspond to lines B, C and D for $\alpha = 9$, 15 and 21 respectively.

The intersections of the straight lines represent the transition points where the curve of the amplitude of the resonance changes its slope. We can also see in figure 3 that these straight lines represent a lower bound for the rate of growth of the amplitude of the resonance. In fact, these straight lines form a convex hull of the amplitude of resonance.

Second example: This example is very similar to the previous one but we include an additional harmonic on the perturbation function:

$$f(x) = \frac{\epsilon}{2\pi}\left(\sin(2\pi x) + 10^{-4}\sin(2\pi(3x)) + 10^{-21}\sin(2\pi(9x))\right) \quad . \tag{28}$$

We want to find now the rate of growth of the amplitude of the resonance for a periodic orbit with rotation number $\frac{4}{9}$. The numerical computation of $\Delta_{4/9}$ is shown in figure 4. In this example we can see three different behaviors of the rates of growth of the amplitude: For small values of ϵ, the curve looks like a segment of a straight line; this part of the curve is dominated by the third harmonic of (28). For values of ϵ in the interval $10^{-4} < \epsilon < 10^{-2}$, the second harmonic of (28) is the dominant one. Finally, for $\epsilon > 10^{-2}$, the dominant harmonic is the first one.

As in the first example, we can find the convex hull (lower bound) for the curve of $\Delta_{4/9}$. This lower bound is given by the amplitude of the resonances obtained from the perturbation function when we consider only one harmonic. In this case we can compute the asymptotic behavior of $\Delta_{4/9}$ taking only one harmonic of the perturbation function (28):

$$\begin{aligned} f_A(x) &= \tfrac{\epsilon}{2\pi}\sin(2\pi x) \quad , \\ f_B(x) &= \tfrac{\epsilon}{2\pi}10^{-4}\sin(2\pi(3x)) \quad , \\ f_C(x) &= \tfrac{\epsilon}{2\pi}10^{-21}\sin(2\pi(9x)) \quad . \end{aligned} \tag{29}$$

Figure 4 shows three straight lines which correspond to the rate of growth of the three harmonics, that is, when we take only one harmonic in the perturbation function. We observe again that the curve which represents $\Delta_{4/9}$ for the perturbation function (28), is located over the three straight lines and the values of ϵ, where the curve changes its slope, can be estimated by the intersection points of the three straight lines.

From the last examples, we can formulate a more general conjecture:

<u>Lower bound conjecture</u>:

Let $\epsilon f(x,\epsilon)$ be an analytic perturbation function. For any rational rotation number $\frac{p}{q}$, we can estimate a lower bound for the amplitude of the resonance with this rotation number in the following form:

1. Using the Fourier series of the perturbation function, we truncate this series at $N \leq q$. Then we must take into account only those harmonics for which the integer q can be written as a linear combination of them.

2. For any set of the selected harmonics, we compute their asymptotic behavior, in this step we can follow the rules that we described in section 3.

3. From each set of harmonics we plot the corresponding straight line in a graph $log(\Delta_{p/q})$ and $log(\epsilon)$. The set of straight lines forms a convex hull and this represent a lower bound for the amplitude of resonance of the actual perturbed twist map.

4. The transition points, where the curve of the amplitude of resonance changes its slope, can be estimated by the intersection of the straight lines.

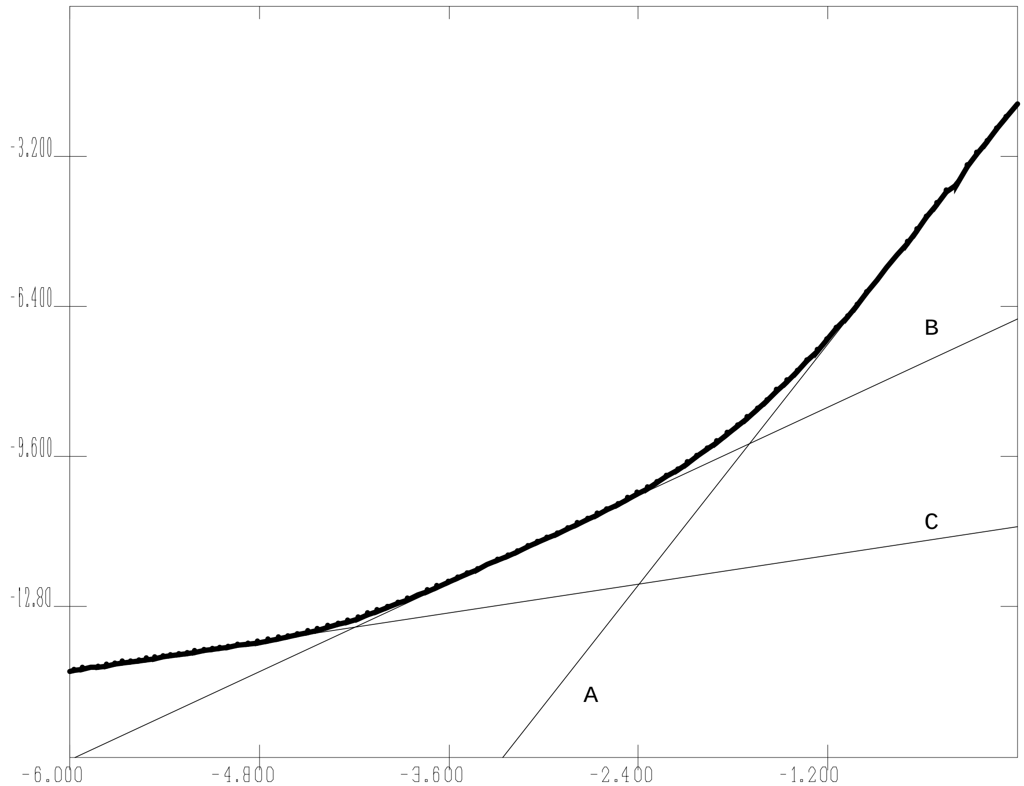

Figure 4. Axes are described in figure 3. The bold line correspond to the value of $\Delta_{4/9}$ using (28). Straight lines A, B and C correspond to the value of $\Delta_{4/9}$ in (28), using only the first, second and third harmonic respectively.

6. Collapse of resonances.

The numerical computation of the amplitudes of resonance for different kinds of perturbation functions gives us the possibility of finding the relation between the value of the perturbation parameter and the amplitude of resonance for any rotation number. Usually, the amplitude of resonance behaves as a monotone function respect to the perturbation parameter. Nevertheless, there is a set of perturbation functions for which the amplitude does not behave as an increasing function in some interval of values of the perturbation parameter; in this interval the amplitude decays to zero and afterwards grows to reach the previous value of the amplitude of resonance. In this section we are going to study this phenomenon and give a simple explanation in terms of the asymptotic behavior of the amplitude of resonance of the different harmonics that belong to the perturbation function. We call this phenomenon the collapse of resonance.

This phenomenon happens when the contributions of two or more harmonics of the perturbation function are out of phase for some small domain of the parameter. In this case the total contribution to the amplitude is null and the size of this resonance goes to zero. We have this situation when two harmonics of the perturbation function have opposite sign and they behave as the principal contribution to the amplitude. We can estimate the range of values of ϵ where we expect to find collapse; this situation takes place when the contribution of the different harmonics of the perturbation function are of the same order of magnitude. The asymptotic behavior of these harmonics corresponds to straight lines in a log-log graph. For any two harmonics which have opposite sign, the

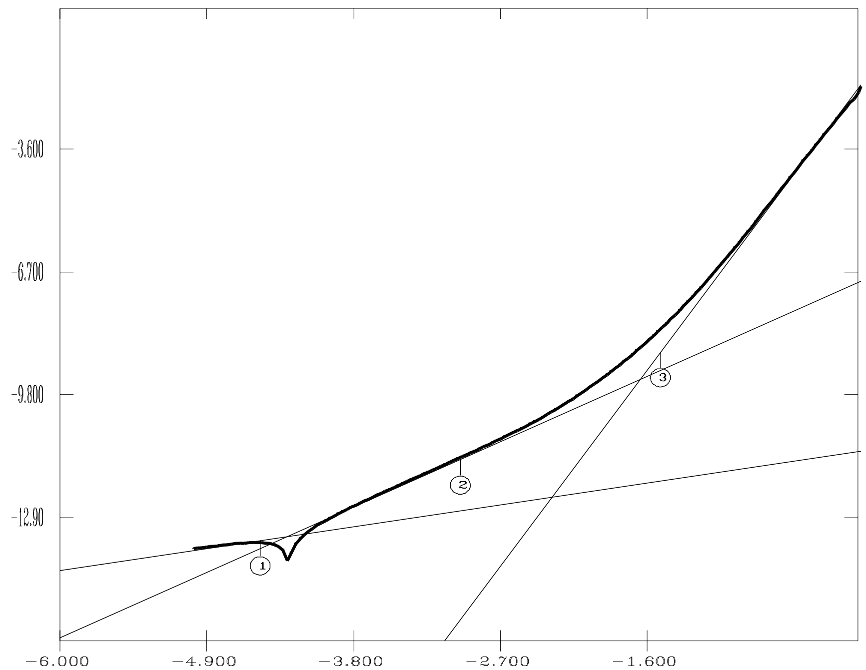

Figure 5. Rotation number : $\frac{4}{9}$ Line 1 Harmonic 9, Line 2 Harmonic 3, Line 3 Harmonic 1.

intersection point of the two lines, represents the value of the parameter where the resonance could collapse.

The next examples show the way in which we can predict the collapse of a specific resonance. Consider the following perturbation function:

$$\epsilon f_1(x) = \frac{\epsilon}{2\pi}\left\{\sin(2\pi x) + 10^{-4}\sin(2\pi(3x)) - 10^{-21}\sin(2\pi(9x))\right\} \quad .$$

Figure 5 shows the numerical computation of the amplitude of resonance $\Delta_{4/9}$, and also the the straight lines corresponding to the asymptotic behavior of each harmonic alone. We can see that the amplitude collapses in the region of parameters where the line 1 (which correspond to the ninth harmonic) and the line 2 (third harmonic) have their intersection point. In this case we have collapse because the ninth and third harmonics have opposite sign and similar magnitude.

Now we take the perturbation function:

$$\epsilon f_2(x) = \frac{\epsilon}{2\pi}\left\{\sin(2\pi x) - 10^{-4}\sin(2\pi(3x)) - 10^{-21}\sin(2\pi(9x))\right\} \quad .$$

The graph of the numerical computation of $\Delta_{4/9}$ is shown in figure 6. We also plot the straight lines which correspond to the asymptotic behavior of the three harmonics. In this case the collapse occurs in the neighborhood of the intersection of the line 2 and line 3 (which correspond to the first harmonic). We can predict this collapse because the first and third harmonics have opposite sign.

The *lower bound conjecture* can be extended, now we can include a statement related to the collapse of resonance:

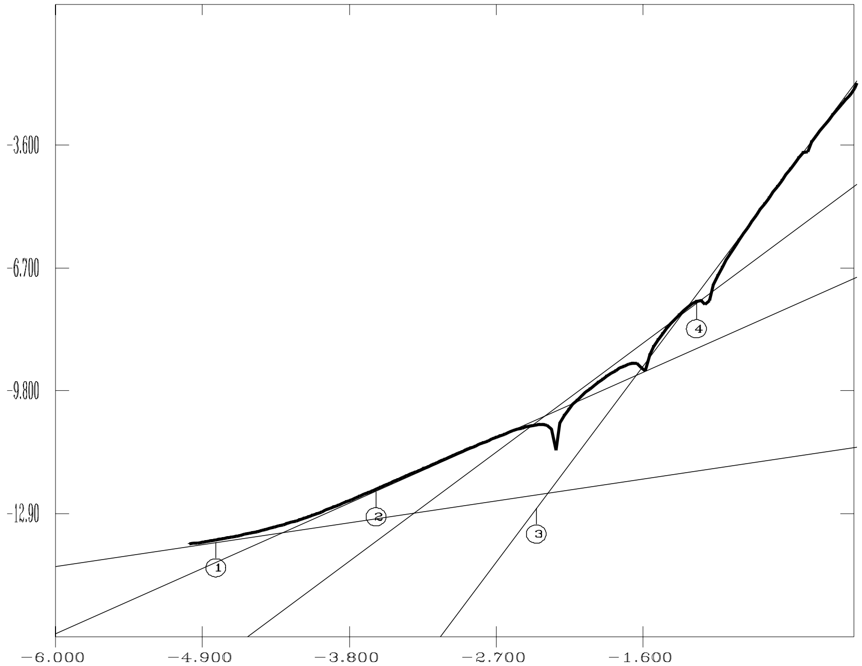

Figure 6. Rotation number : $\frac{4}{9}$ Line 2 Harmonic 3, Line 1 Harmonic 9, Line 3 Harmonic 1.

Collapse of resonance conjecture:

1. The intersection point of two straight lines, which correspond to harmonics with opposite sign, gives the value of the parameter where we expect to have collapse of resonance.

An important fact related to the collapse of resonance phenomenon is that the left branch of the amplitude of resonance, respect to the point where the amplitude has collapsed, corresponds to a periodic orbit which is different from that periodic orbit which is produced by the right branch of the curve. From the dynamical point of view, the collapse of resonance is related to one periodic orbit whose amplitude of resonance is decreasing in some interval of the parameter; the amplitude decreases until the resonance disappears: this is because the hyperbolic periodic orbit collides with the elliptic orbit that shares the same rotation number, this phenomenon corresponds to a saddle–node bifurcation which in turn corresponds to a creation or destruction of a set of periodic orbits. Simultaneously, a pair of periodic orbits, with the same rotation number, is created by the mechanism of the saddle–node bifurcation, the amplitude of resonance of the corresponding hyperbolic periodic orbit grows from zero and it behaves like a step function. Birkhoff twist theorem shows that any twist map has at least one pair of periodic orbits for any rational rotation number which belongs to the rotation interval of the map. Usually, we only have one pair of periodic orbits for any value of the perturbation parameter. The collapse of resonance is an example where we can find two pairs of periodic orbits that share the same rotation number and exist simultaneously for a small open set of values of the perturbation parameter.

7. Example of analytical perturbation.

In this section we show an example where we compute the amplitude of resonance for an analytic perturbation function. Consider the following perturbation function, $\epsilon f(x) = \epsilon \sin(a \cos(2\pi x))$. We can see that $f(x)$ is an analytic periodic function and the average of this function is 0. Suppose that we want to study the resonance 3/10 of this problem in a range of values of the perturbation parameter $\epsilon \in [0, 1]$. We fix the parameter $a = 0.01$.

The first step is to find the Fourier expansion of the perturbation function. It is easy to get the expansion using some properties of Bessel functions [1]:

$$\sin(a \cos(x)) = 2 \sum_{k=0}^{\infty} J_{2k+1}(a) \sin(2\pi(2k+1)x) \quad , \tag{30}$$

where $J_i(a)$ are the Bessel functions of integer order i. Because we are interested in the resonance 3/10, we must cut the Fourier series at harmonic 10:

$$\epsilon f(x) = 2\epsilon \sum_{k=0}^{4} J_{2k+1}(a) \sin(2\pi(2k+1)x) \quad . \tag{31}$$

We only have five harmonics, 1, 3, 5, 7 and 9, and we can find five linear combinations of these harmonics in order to reach the harmonic 10. The next step is to determine the asymptotic behavior of each linear combination. The results are given in the following table:

Harmonics	$\bar{m}$	$A_{3/10}^{\text{asympt}}$
1 and 9	1	3.6164×10^{-10}
3 and 7	1	3.2794×10^{-9}
5	1	1.0384×10^{-9}
1	10	1.2244×10^{-7}
1, 3 and 5	4	1.5325×10^{-7}

We can verify that the coefficients of the perturbation function (31) are positive, that is $J_i(a) > 0$ for $a = 0.01$. This means that we do not expect to have any collapse of resonance, in this case the amplitude of resonance must behave as a monotone function. Figure 7 shows the numerical computation of amplitude of resonance $\Delta_{3/10}$ with perturbation function (30). The five straight lines correspond to the asymptotic behavior of the harmonics given above. This set of straight lines determines a good estimate of $\Delta_{3/10}$. The range of values of the perturbation parameter in our example is $\epsilon \in (0, 3.163)$

It is important to see that we have obtained a good estimate of $\Delta_{3/10}$ using only asymptotic information of the linear combination of five harmonics and these asymptotic behaviors were computed for values of the parameter close to zero.

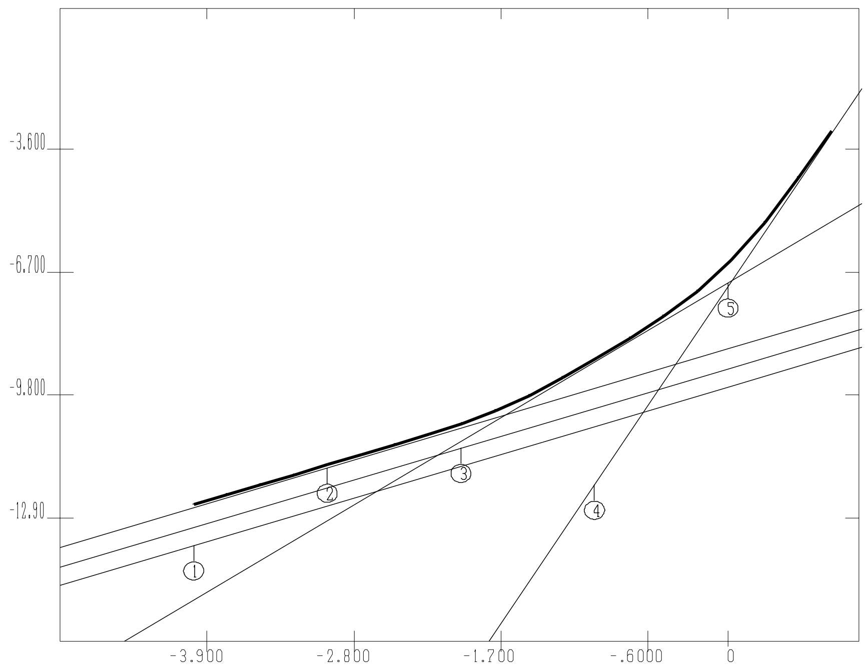

Figure 7. Rotation number : $\frac{3}{10}$

Line 1	Harmonics 1 and 9,	Line 2	Harmonics 3 and 7,
Line 3	Harmonic 5,	Line 4	Harmonic 1,
Line 5	Harmonics 1, 3 and 5.		

8. Conclusion.

We have shown a simple algorithm to study the nonlinear behavior of any resonance of the twist maps. We have presented many examples where we could predict the amplitude of resonance given any rational rotation number. In all the examples, we only needed to know the asymptotic behavior of the standard map and we used a simple perturbation function which has only few harmonic terms of the Fourier expansion of the initial perturbation function. The most difficult task was to determine the eigenvalue of the hyperbolic periodic orbit with the same rotation number when the value of the parameter is asymptotically small. Using this information, we could obtain a good set of lower bounds for the amplitude of resonances and we could predict the collapse of resonances.

It is important to remark that using only linear information for small values of the parameter we were able to predict the nonlinear behavior of the map when the perturbation parameter is of order 1.

The sets of perturbation functions that we have used in our previous examples are written as the sum of sine functions. The main reason to use only odd function is related to our numerical method to compute periodic orbits. When the General Standard Map is written with an odd perturbation function, this map can be described as the composition of two maps, each map is an involution and it has a pair of invariant curves in the plane. It is possible to show that we can find periodic orbits which have points in these invariant curves, therefore we only need to look for the periodic orbits, for any rotation number, on the symmetry lines. Using this method, we can simplify significantly the numerical procedure to find periodic orbits because we are able to find these orbits using a one-dimensional root-finder. Nevertheless, the method to estimate the amplitude of resonance can be used for any perturbation function.

Acknowledgment.

The author is grateful to Dr. Carles Simó for helpful discussions and comments on the manuscripts and also for the computation facilities. The author is also grateful to Ana Cecilia Perez for the computer facilities in our laboratory. This work was supported by Grant CONACyT G25427-E.

Appendix 1.

Small perturbations of the integrable Hamiltonian system gives rise to more complex dynamics in phase space. A suitable way to perturb an integrable twist map is using a generating function; in this way we can assure that the transformed map has the same symplectic structure. Any integrable twist map can be written in the following form:

$$\begin{aligned} r_{i+1} &= r_i \ , \\ \phi_{i+1} &= \phi_i + \alpha(r_{i+1}) \ , \end{aligned} \tag{32}$$

where $(\phi_i, r_i) \in \mathsf{S}^1 \times \mathrm{IR}$, $i \in \mathbb{Z}$. In this case the value of the function $\alpha(r)$ coincides with the rotation number of the orbit and is a monotone function; from now on we set $d\alpha(r)/dr > 0$. The generating function of the map (32) is given by the expression:

$$G(r_{i+1}, \phi_i) = r_{i+1}\phi_i + B(r_{i+1}) \ , \tag{33}$$

where $\frac{dB(r)}{dr} = \alpha(r)$. Let $g(r, \phi, \epsilon)$ be an analytic function which is periodic with respect to ϕ. Then we can perturb (33) in the following form:

$$G(r_{i+1}, \phi_i) = r_{i+1}\phi_i + B(r_{i+1}) + \epsilon g(r_{i+1}, \phi_i, \epsilon) \ . \tag{34}$$

Define the functions $f_1(r, \phi, \epsilon) = \frac{\partial g}{\partial \phi}$ and $f_2(r, \phi, \epsilon) = \frac{\partial g}{\partial r}$, then our perturbed twist map if given in the form:

$$\begin{aligned} r_{i+1} &= r_i + \epsilon f_1(r_{i+1}, \phi_i, \epsilon) \ , \\ \phi_{i+1} &= \phi_i + \alpha(r_{i+1}) + \epsilon f_2(r_{i+1}, \phi_i, \epsilon) \ . \end{aligned} \tag{35}$$

This is a one parameter family of twist maps, the parameter is ϵ and we assume that $|\epsilon| << 1$. The functions f_1 and f_2 are periodic respect the angular coordinate ϕ and we impose $lim_{\epsilon \to 0} \epsilon f_i(r, \phi, \epsilon) = 0$.

In order to study the dynamics around monotone periodic orbits of the perturbed map (35) it is convenient to transform it. The idea is to find a simple map which coincides with (35) in the neighborhood of specific rotation number. This transformation must be symplectic in order to preserve the Hamiltonian properties of the map (35). A way of carrying out symplectic transformation is to find a suitable generating function which can be used to construct the transformation. We want to transform the set of coordinates (r, ϕ) into new coordinates (ρ, ϕ) with the following property: $\rho_{i+1} = \alpha(r_{i+1}) + \epsilon f_2(r_{i+1}, \phi_i, \epsilon)$.

Using the implicit function theorem we can write r_{i+1} as a function of ϕ_i and ρ_{i+1}, then we obtain $r_{i+1} = \gamma_1(\rho_{i+1}, \phi_i, \epsilon)$. Because ϵ is small, this function is then written as:

$$r_{i+1} = \alpha^{-1}(\rho_{i+1}) + \epsilon \gamma_2(\rho_{i+1}, \phi_i, \epsilon) \ . \tag{36}$$

Therefore our generating function is given by:

$$G(\rho_{i+1},\phi_i) = \alpha^{-1}(\rho_{i+1})\phi_i + \epsilon \int_{\phi_i} \gamma_2(\rho_{i+1}, s, \epsilon) ds \quad .$$

Because the second equation of (35) is transformed to $\phi_{i+1} = \phi_i + \rho_{i+1}$, then the variable ϕ_i can be substituted in (36):

$$r_{i+1} = \alpha^{-1}(\rho_{i+1}) + \epsilon\gamma_2(\rho_{i+1}, \phi_{i+1} - \rho_{i+1}, \epsilon) = \alpha^{-1}(\rho_{i+1}) + \epsilon\hat{\gamma}_2(\rho_{i+1}, \phi_{i+1}, \epsilon) \quad . \tag{37}$$

We see that the relation (37) involves only coordinates with subindex $i+1$, then we can shift index $i+1$ to i. We can rewrite the first equation of (35), using relation (37) in the following form:

$$\alpha^{-1}\rho_{i+1} + \epsilon\hat{\gamma}_2(\rho_{i+1}, \phi_{i+1}, \epsilon) = \alpha^{-1}\rho_i + \epsilon\hat{\gamma}_2(\rho_i, \phi_i, \epsilon) + \epsilon f_2(\rho_i + \epsilon\hat{\gamma}_2(\rho_i, \phi_i, \epsilon), \phi_i, \epsilon) \quad .$$

Replacing ϕ_{i+1} by $\phi_i + \rho_{i+1}$ and using again the implicit function theorem, we can write ρ_{i+1} as a function of ρ_i and ϕ_i. Therefore the new map is then given by the following expression:

$$\begin{aligned} \rho_{i+1} = &\quad \rho_i + \epsilon\hat{\gamma}_3(\rho_i, \phi_i, \epsilon) \quad , \\ \phi_{i+1} = &\quad \phi_i + \rho_{i+1} \quad . \end{aligned} \tag{38}$$

The map (38) must be a symplectic map, this means that the determinant of the Jacobian matrix of (38) must be one. This condition implies that the function $\hat{\gamma}_3(\rho_i, \phi_i, \epsilon)$ does not depend on ρ_i. The final form of map (38) can be written then in this form:

$$\begin{aligned} \rho_{i+1} = &\quad \rho + \epsilon V(\phi_i, \epsilon) \quad , \\ \phi_{i+1} = &\quad \phi_i + \rho_{i+1} \quad . \end{aligned} \tag{39}$$

We denote (39) as the General Standard Map. We must remark that the angular coordinate ϕ was not transformed from (35) to (39), so the function $V(\phi,\epsilon)$ remains periodic with respect to the first argument, $V(\phi+1,\epsilon) = V(\phi,\epsilon)$.

The map (39) could not be a global representation of the dynamics of the map (35) because the domain of transformation (38) could be an open set of $\mathbb{R}$. The map (39) can be then used around a strip of the cylinder $\mathsf{S}^1 \times \mathbb{R}$. Therefore in order to study monotone periodic orbits we can choose the Standard Map as a general representation of a twist symplectic map.

Appendix 2.

Let us consider the standard map (1); for small values of the parameter ϵ we can estimate the eigenvalues of any monotone periodic orbit with rotation number $\frac{p}{q}$. Suppose that $V(x)$ is a $\mathcal{C}^k$ periodic function with $k \geq 2$. The map F^q can be written in the neighborhood of periodic orbit with first order approximation [11]:

$$
\begin{aligned}
y_q = & \quad y_1 + \epsilon \sum_{i=1}^{q-1} V(x_i + iy_i) + \mathcal{O}(\epsilon^2) \quad , \\
x_q = & \quad x_1 + qy_1 + \epsilon \sum_{i=1}^{q-1} (q-i) V(x_i + iy_i) + \mathcal{O}(\epsilon^2) \quad .
\end{aligned}
$$

Using the coordinate transformation $y = \frac{p}{q} + \epsilon^{1/2}\zeta$, the previous map takes the form:

$$
\begin{aligned}
\zeta_q = & \quad \zeta_1 + \epsilon^{1/2} \sum_{i=1}^{q-1} V(x_i + iy_i) + \mathcal{O}(\epsilon) \quad , \\
x_q = & \quad x_1 + \epsilon^{1/2} q\zeta_1 + p + \mathcal{O}(\epsilon) \quad .
\end{aligned}
$$

The eigenvalues of the fixed points $x_1 = x_q - p$ and $\zeta_q = \zeta_1$ are given by:

$$
\lambda_{\pm} = 1 \pm \epsilon^{1/2} \left(\sum_{i=1}^{q-1} DV(x_i + i\frac{p}{q}) \right)^{1/2} + \mathcal{O}(\epsilon) \quad ,
$$

where DV is the derivate of V and the set of values $\{x_i\}$ must be the solution of the following equation:

$$
\sum_{i=1}^{q-1} V(x_i + i\frac{p}{q}) = 0 \quad .
$$

It is generic that the asymptotic value of the eigenvalues satisfies

$$
|\lambda_{\pm} - 1| \sim \mathcal{O}(\epsilon^{1/2}) \quad ,
$$

therefore, the amplitude of resonance is given by the following relation obtained from the pendulum equation:

$$
\Delta_{p/q} = \frac{\epsilon^{1/2}}{q} \sqrt{\frac{\sum_{i=1}^{q-1} DV(x_i + i\frac{p}{q})}{2\pi}} + \mathcal{O}(\epsilon) \quad .
$$

However, sometimes there are no solutions of the equation $\sum_{i=1}^{q-1} V(x_i + i\frac{p}{q}) = 0$ and the eigenvalues depend on values higher than $1/2$ of the exponent of ϵ. This happens when the Fourier expansion of $V(x)$ has a null coefficient in the n-th harmonic term. In this case the amplitude of resonance depends on the other harmonics of the perturbation function and we cannot compute this amplitude of resonance in a straightforward manner using the first order approximation of our map.

Appendix 3.

In this appendix we are going to sketch a different procedure to determine a normal form around monotone periodic orbits of a general standard map (4). In this case we choose V as an antisymmetric function but it is possible to carry out our computation without this assumption. Details of developments to obtain these resonant normal forms are given in [9].

The procedure to obtain the resonant normal form of F^n is divided in four steps:

1. Given a rational rotation number $\frac{p}{n}$ we compute an explicit form of the n-time mapping of F, such that: $(x_n, y_n) = F^n(x_0, y_0)$.

2. Obtain the root of this map solving the following equation:

$$F^n(x,y) - (x,y) - (p,0) = 0 \quad . \tag{40}$$

   The set of solutions of this equation corresponds to a monotone periodic orbit with rotation vector $\frac{p}{n}$.

3. Starting with a scalar transformation $y = \epsilon\zeta_{\{0\}} + \frac{p}{n}$ we can find a set of maps F^n_m, such that the difference between the map F^n_m and the identity is of order $\mathcal{O}(\epsilon^m)$. For each scale transformation, $\zeta_{\{m\}} = \epsilon\zeta_{\{m+1\}} + \phi_{\{m+1\}}(x)$ where $\zeta_{\{m+1\}}$ is the new axial coordinate, $\phi_{\{m+1\}}(x)$ is a periodic function of the angular coordinate x, the new map is closer to the identity map than the previous one, the distance is of order $\mathcal{O}(\epsilon^{m+1})$.

4. The lower order term of the new map, after m transformations, is of order $\mathcal{O}(\epsilon^m)$. Consider only the lower order terms, these terms are polynomials of the axial coordinate $\zeta_{\{m\}}$ whose coefficients are periodic functions of the angular variable x. The fixed points of this map correspond to monotone periodic orbits of the initial map. The fixed points are obtained when $\zeta_{\{m\}} = 0$, therefore the maximum number of fixed points is related to the maximum harmonic of the periodic term which depends on x, the number of fixed points is equal to the period of the corresponding monotone periodic orbit of the initial map. The set of monotone periodic orbits is well ordered respect to the rotation number, the Aubry-Mather theory shows that there exist monotone periodic orbits for any rational rotation number that belongs to the rotation interval of the twist map [7]. If the maximum

harmonic is less than n, where n is the period of the monotone periodic orbit which we want to find, then it is impossible to obtain a periodic orbit with period m. This means that the periodic term of order $\mathcal{O}(\epsilon^m)$ which does not depend on $\zeta_{\{m\}}$ must be null otherwise we can obtain a monotone periodic orbit located in a wrong order. Therefore we can find the next scale transformation where the lower order terms are of order $m+1$. We can repeat the procedure until the maximum harmonic of the periodic terms, which depend on x, is equal to n. We denote by S_n the number of scale transformations that we can perform until we get the maximum harmonic equal to n.

After performing S_n scale transformations, we obtain a map which looks like a time ϵ^{S_n} flow of a Hamiltonian system, this system is a chain of n-pendula, the phase space of this system looks like the picture shown in figure 1. The differential equation of this Hamiltonian system is given in equation (15). From this equation, we can estimate the amplitude of resonance $\Delta_{p/q}$, this amplitude is given in equation (16).

Now we show the procedure to compute F^n. The first step is to rewrite the equation (4) as a second order difference equation:

$$x_{i+2} = 2x_{i+1} - x_i + \sum_{j=1}^{N} \epsilon^{a_j} c_j \sin(2\pi(jx_{i+1})) \quad , \tag{41}$$

where the function $V(x)$ was substituted by equation (3), in this case we rewrite the perturbation function in terms of trigonometric series.

Using basic properties of Bessel functions it is possible to obtain the n-th iteration of the angular variable, x_n, in terms of the initial values x_0 and x_1. Taking this relation, we obtain the map F^n:

$$x_n = nx_1 - (n-1)x_0 + \sum_{i=0}^{n-2} (n-i-1) P_i(x_0, x_1) \quad , \tag{42}$$

where

$$P_m(x,\bar{x}) = \sum_{i=1}^{N} \epsilon^{d_j} q_j \sum_{\substack{l_b^a \in \mathbb{Z} \\ a=1,\dots,m \\ b=1,\dots,N}} \left\{ \prod_{s=1}^{N} \prod_{t=1}^{m} J_{l_s^t}\left(\epsilon^{d_s} q_s S_{N,i}^{t}\right) \right\} \times$$
$$\sin\left(S_{N,i}^{m+1}\bar{x} - S_{N,i}^{m} x\right) \quad ,$$

where $J_i(x)$ are the Bessel functions of integer order i and the coefficients $S_{\alpha,i}^{\beta}$ are defined by:

$$S_{\alpha,i}^{\beta} = 2\pi \sum_{r=0}^{\beta-1} (\beta - r) \sum_{j=1}^{\alpha} j l_j^r \quad ,$$

the set of numbers l_j^r are integer numbers with the restriction $l_j^0 = \delta_j^i$, where δ_j^i is the Kronecker delta function.

Now, it is convenient to write the second order difference equation (42) as two difference equations, where we define $y_{i+1} = x_{i+1} - x_i$, for $i = 0, \ldots, n$. Then we obtain the following set of equations:

$$\begin{aligned} y_n = & \quad y_0 + \sum_{j=0}^{n-1} P_j(x_0 - y_0, x_0) \quad , \\ x_n = & \quad x_0 + n y_0 + \sum_{j=0}^{n-1} (n-j) P_j(x_0 - y_0, x_0) \quad , \end{aligned} \tag{43}$$

The map (43) can be written as a power series of the perturbation parameter ϵ:

$$\begin{aligned} y_n = & \quad y_0 + \sum_{j=0}^{n-1} \sum_{s \in R_N} \epsilon^s G_j^s(x_0, y_0) \quad , \\ x_n = & \quad x_0 + n y_0 + \sum_{j=0}^{n-1} (n-j) \sum_{s \in R_N} \epsilon^s G_j^s(x_0, y_0) \quad , \end{aligned} \tag{44}$$

the functions $G_j^s(x, y)$ have the following definition:

$$G_j^s(x, y) = \sum_{A(j,N,i,s)} \mathcal{G}_{(j,N,i,s)} \sin\left(\bar{S}_{N,i}^{j+1} x + S_{N,i}^j y\right) \quad . \tag{45}$$

The coefficients $\mathcal{G}_{(j,N,i,s)}$ are constant rational numbers. The sets of index R_N and $A(j, N, i, s)$ define the domain of the index s, l_a^b, i and t_c in the form:

$$R_N = \left\{ s \in \mathbb{Z} \quad \text{such that} \quad s = \sum_{t=1}^{N} w_t a_t \quad \text{and} \quad w_t \in \mathbb{Z}^+ \right\}$$

the set of numbers a_t are the exponents of the perturbation parameter in equation (3). The set $A_{(j,N,i,s)}$ defines the domain of values of the sets of integer numbers $\{l_r^b\}$, where $l_r^b \in \mathbb{Z}$, $r = 1, \ldots, m$ and $b = 1, \ldots, N$, the set $\{i\}$ such that $i = 1, \ldots, N$ and the set $\{t_c\}$ where $t_c \in \mathbb{Z}^+$. These sets of integer numbers have the following restriction:

$$\left[a_i + \left(\sum_{k=1}^{N} \sum_{b=1}^{m} a_b |l_k^b| \right) + \sum_{r=1}^{N} 2 t_r a_r \right] = s \quad . \tag{46}$$

The coefficient $\bar{S}_{\alpha,i}^{\beta}$ are defined by:

$$\bar{S}_{\alpha,i}^{\beta} = 2\pi \sum_{r=0}^{\beta-1} \sum_{j=1}^{\alpha} j l_j^r \quad .$$

We can check that for a fixed value of the integer s the domain of index l_r^b is bounded, therefore the maximum value of the coefficients $\bar{S}_{N,i}^{j+1}$ and $S_{N,i}^{j}$ are also bounded.

The map (44) can be understood as a power series in ϵ where the coefficients are a finite Fourier series of coordinates x and y. The maximum harmonic of any of these Fourier series corresponds to the maximum value of $\bar{S}_{N,i}^{j+1}$ and $S_{N,i}^{j}$.

Bibliography.

[1] M. Abramowitz and I. Stegun, Handbook of Mathematical Functions (Dover, New York, 1972).

[2] B. V. Chirikov, Phys. Report **52**, (1979), 262.

[3] J. M. Greene, J. of Math. Phys. **20**, 1183 (1979).

[4] D. F. Escande, Physica D **6**, (1982), 119.

[5] A. J. Lichtenberg and M. A. Lieberman, Regular and Stochastic Motion (Springer-Verlag, New York, 1983).

[6] R. S. Mackay, J. D. Meiss and I. C. Persival, Physica D **13**, (1984), 55.

[7] J. M. Mather, Topology **21**, (1982), 457.

[8] A. Olvera and C. Simó, Physica D **26**, (1987), 181.

[9] A. Olvera and C. Simó, Normal forms on a resonant strip for the twist maps. Preprint, (1995)

[10] C. Simó. Analytical and numerical computation of invariant manifold In D. Benest et C. Froeschlé, editors, *Modern Methods in Celestial Mechanics*, Editions Frontières, 1990, 285.

[11] Veerman, P., Dynamical systems and twist maps, *Centre de Reserca Matematica*, UAB, Bellaterra (1993)